Recitation 5

Minimum Spanning Trees

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How’s everyone doing?

Agenda

MST Overview

Prim's Algorithm

Kruskal's Algorithm

MST Properties

MST Practice Problem

Overview

Definitions:

- A **spanning tree** T of a graph $G = (V, E)$ is an acyclic subset of G's edges that connects all vertices of G
- T is a **minimum spanning tree** of a connected weighted graph $G = (V, E, w)$ if its weight $w(T) = \sum_{e \in T} w(e)$ is minimized

Overview

General Procedure:

1. $T \leftarrow \varnothing$
2. **while** T is not spanning **do**

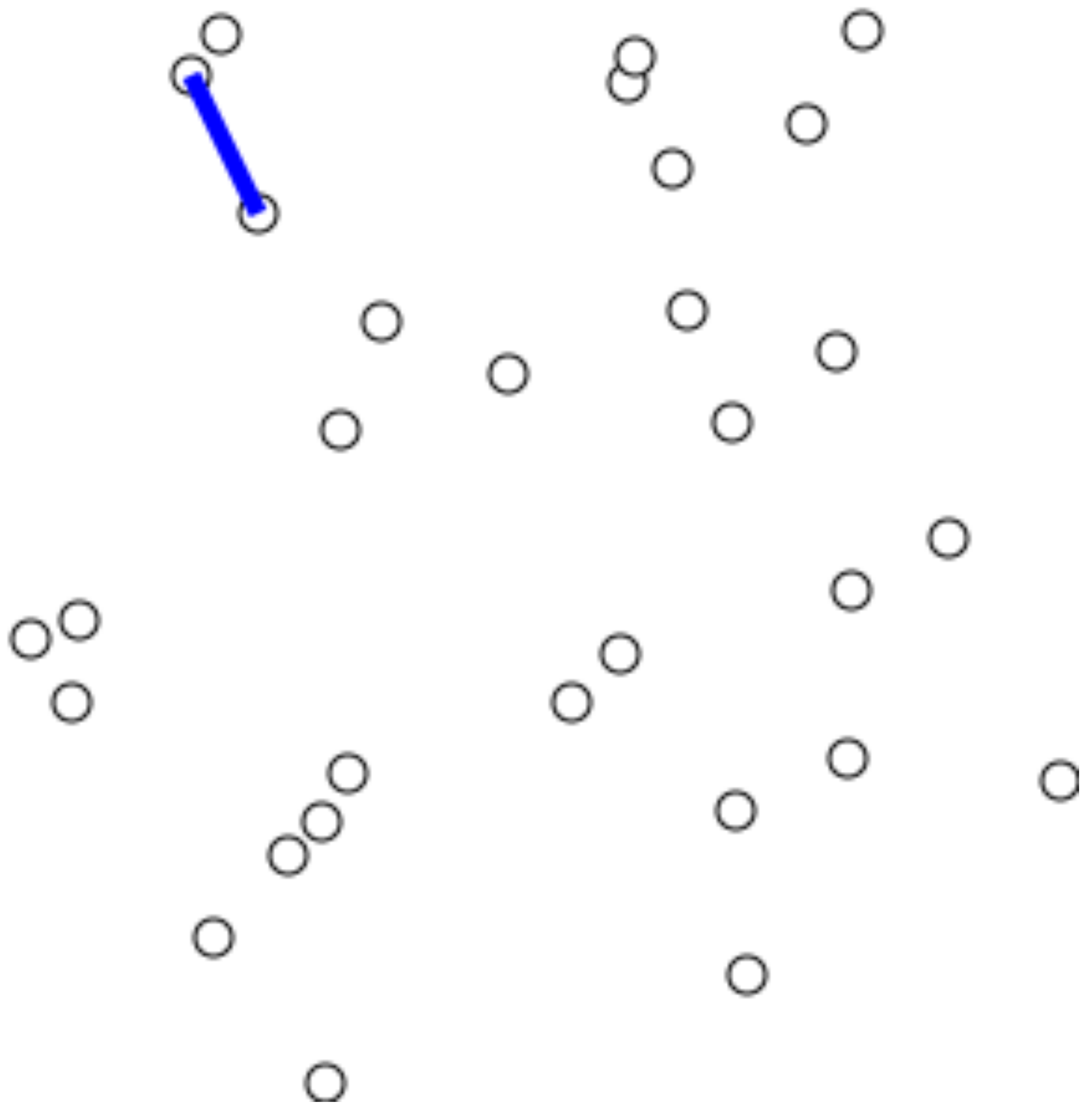
   Add a "safe" edge e to T such that T remains a subset of some MST
3. **return** T

How do we find safe edges?

Prim's Algorithm

Overview

Idea: Greedily select the cheapest edge that connects the tree to a new vertex.



Prim's algorithm applied to points in a Euclidean plane

Prim's Algorithm

Algorithm

Procedure:

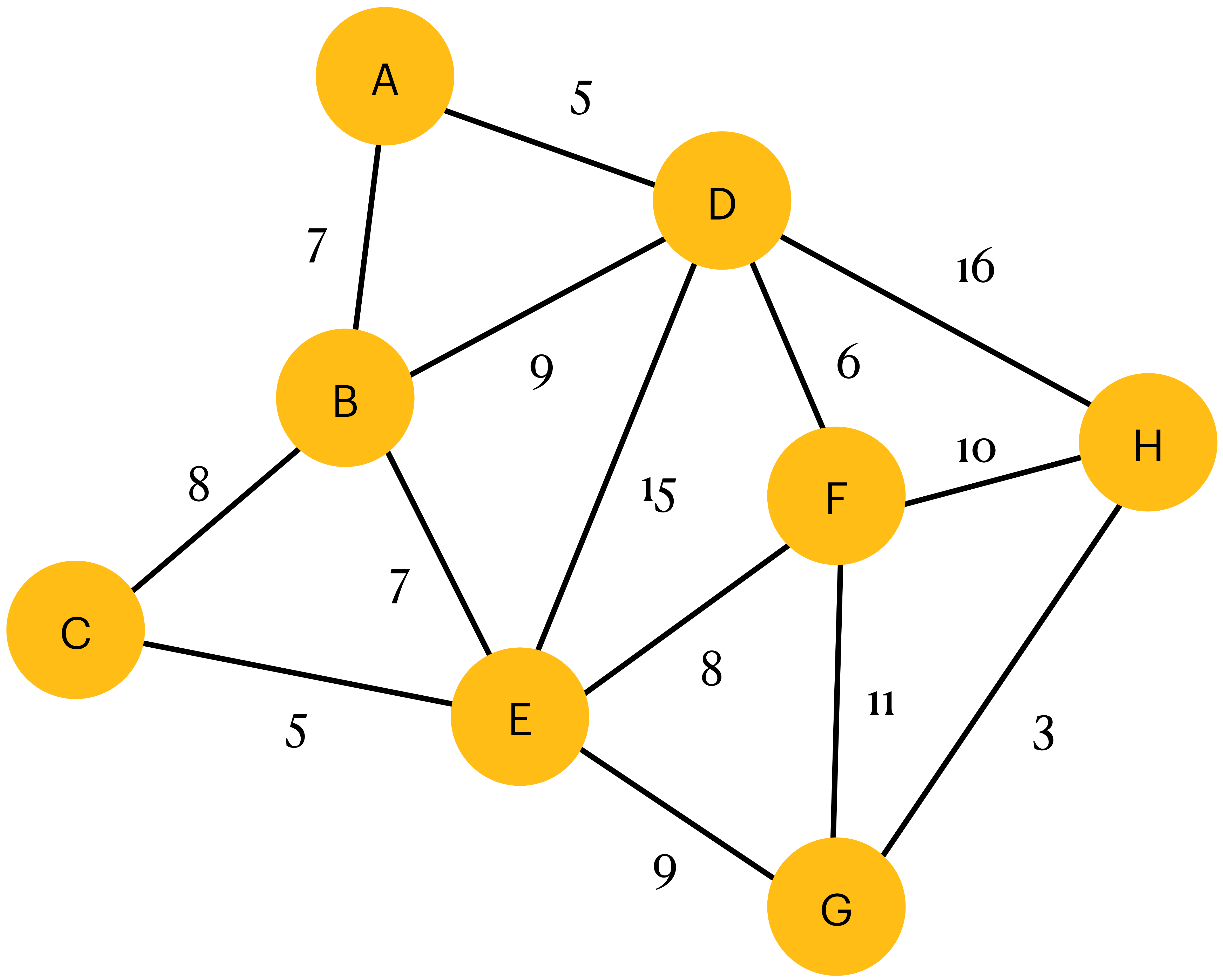
1. Choose an arbitrary starting point $s \in V$
2. $T_V \leftarrow \{s\}$
3. $T_E \leftarrow \emptyset$
4. **while** T_E is not spanning **do**
   A. Find the lowest-weight edge $e = (u, v)$ such that $u \in T_V$ and $v \notin T_V$
   B. $T_V \leftarrow T_V \cup \{v\}$
   C. $T_E \leftarrow T_E \cup \{e\}$
5. **return** T_E

Prim's Algorithm

Correctness

Claim: Prim's algorithm always adds safe edges.

Aside: Review of Cuts



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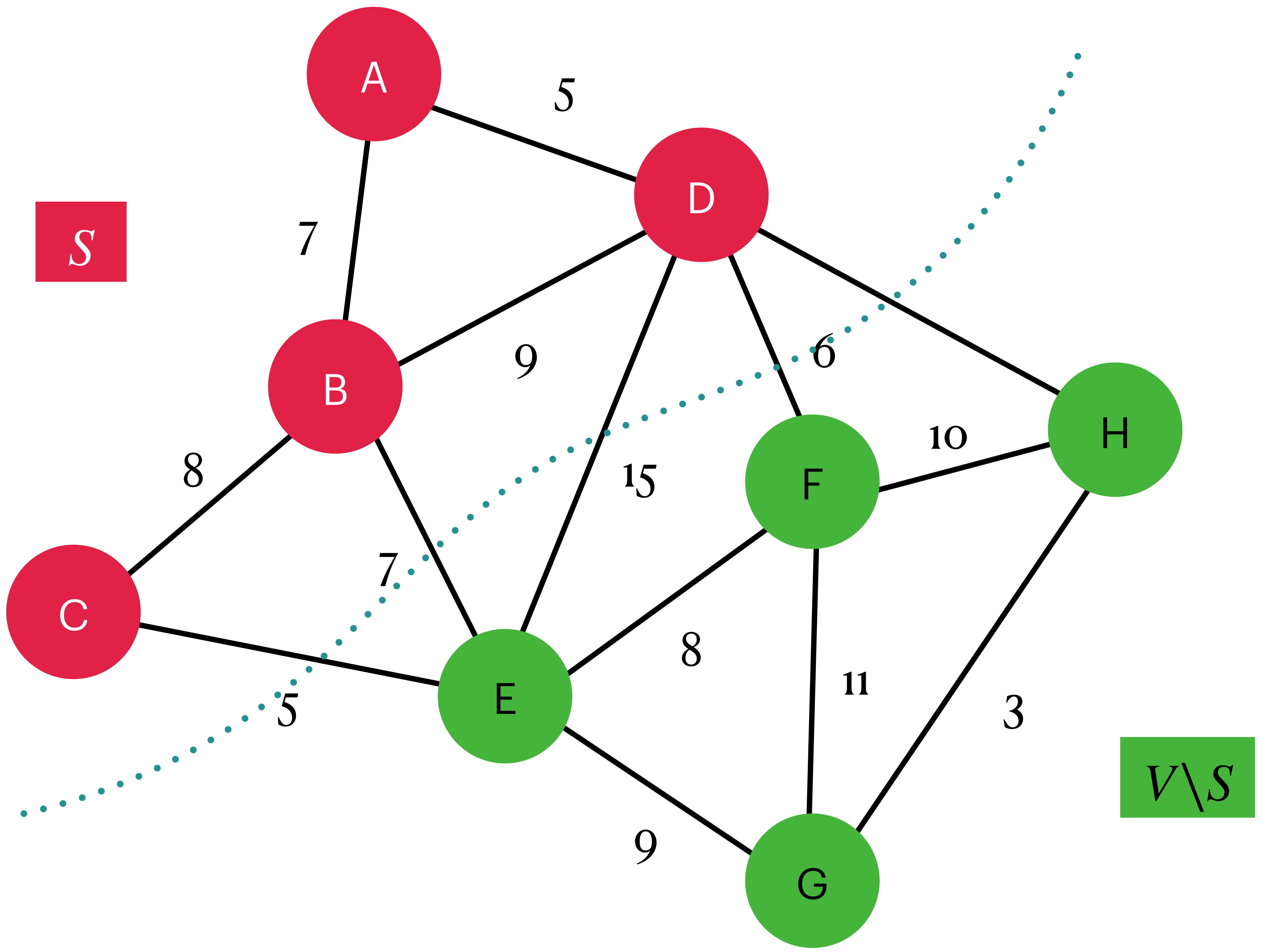

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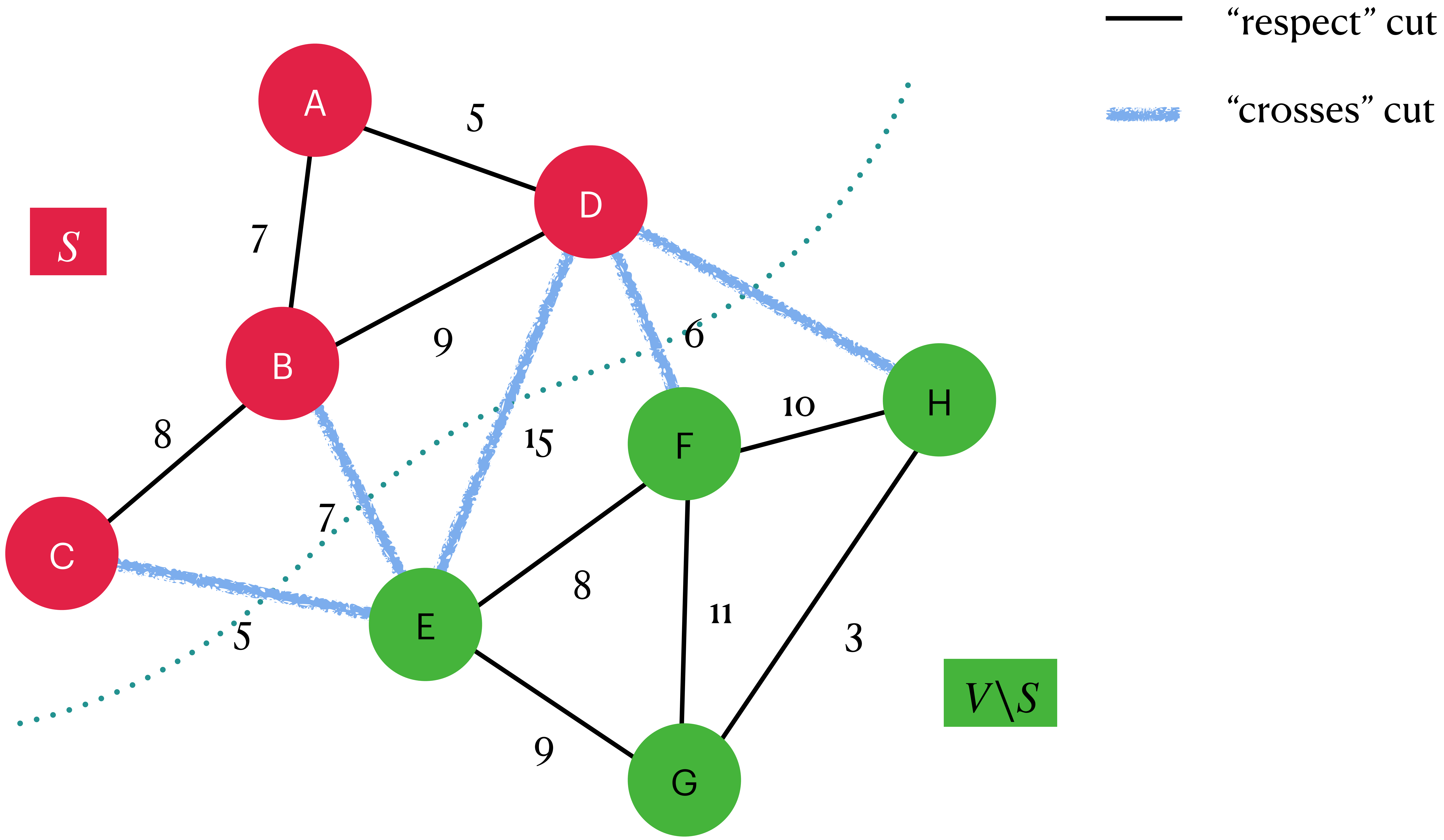

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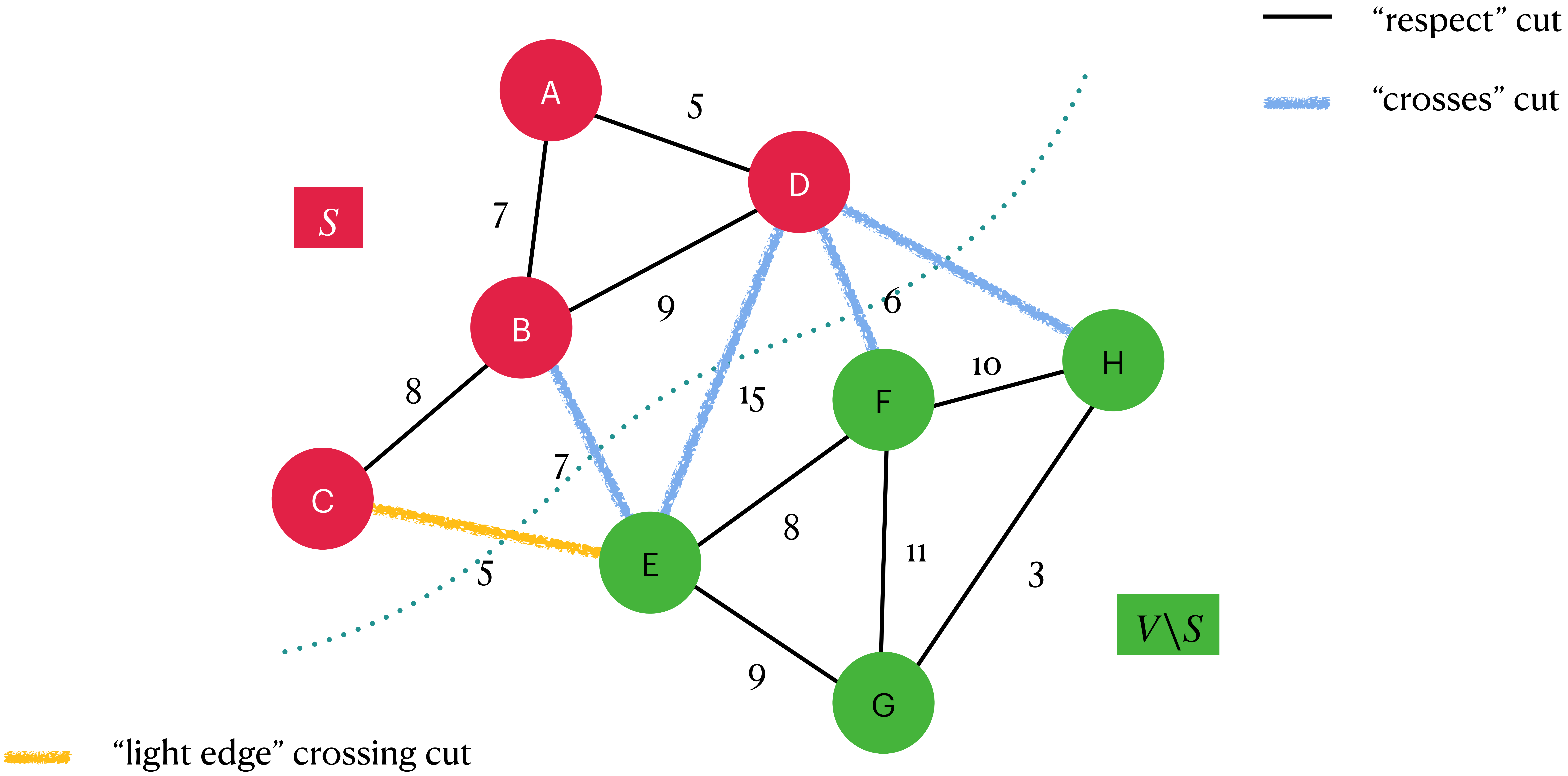


Prim's Algorithm

Correctness

Claim: Prim's algorithm always adds safe edges.

We use **CLRS4 Theorem 31.1:** Given

1. a connected weighted graph $G = (V, E, w)$;
2. a set of edges $A \subseteq E$ belonging to some MST for G;
3. a cut $(S, V \backslash S)$ of G that respects A; and
4. a light edge e crossing the cut,

then e is a safe edge for A.

Prim's Algorithm

Correctness

Consider iteration k:

- Take $A = T_E$ and the cut to be $(T_V, V\backslash T_V)$
- Let $e_k = (u, v)$ be the cheapest edge such that $u \in T_V$ and $v \notin T_V$

We apply **CLRS4 Theorem 31.1**:

1. G is given as a connected weighted graph
2. Assuming only the addition of safe edges to T_E in past iterations, T_E included in some MST for G
3. Cut $(T_V, V\backslash T_V)$ resects T_E because all edges in T_E are between vertices in T_V
4. By definition, e_k is a light edge crossing the cut

Hence e_k is a safe edge.

Prim's Algorithm

Runtime Analysis

Procedure:

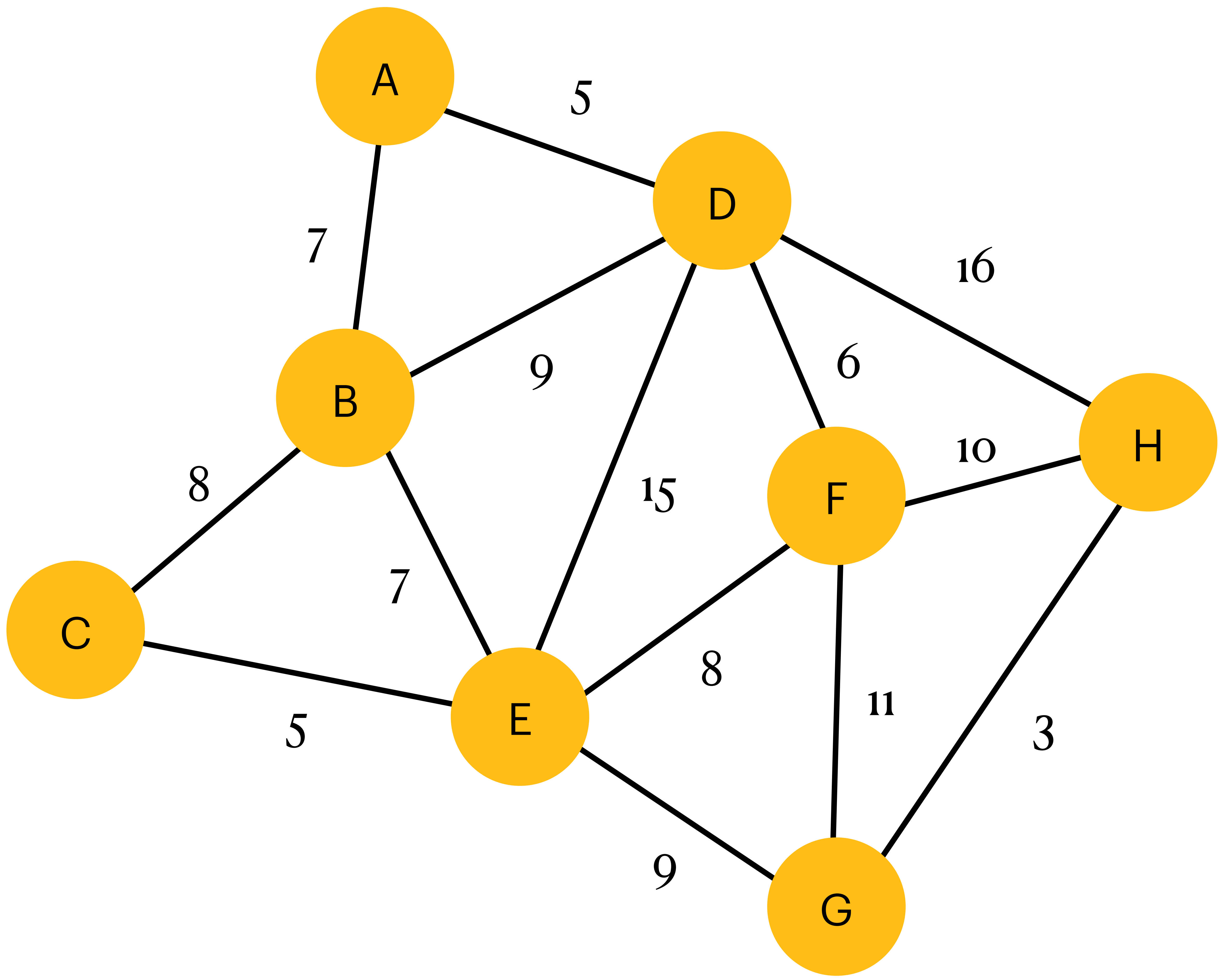
1. Choose an arbitrary starting point $s \in V$
2. $T_V \leftarrow \{s\}$
3. $T_E \leftarrow \varnothing$
4. **while** T_E is not spanning **do** $O(|V|)$
   A. Find the lowest-weight edge $e = (u, v)$ such that $u \in T_V$ and $v \notin T_V$ $O(|E|)$
   B. $T_V \leftarrow T_V \cup \{v\}$
   C. $T_E \leftarrow T_E \cup \{e\}$
5. **return** T_E

What's the runtime?

Naively, it's $O(|V||E|)$—how do we improve it?

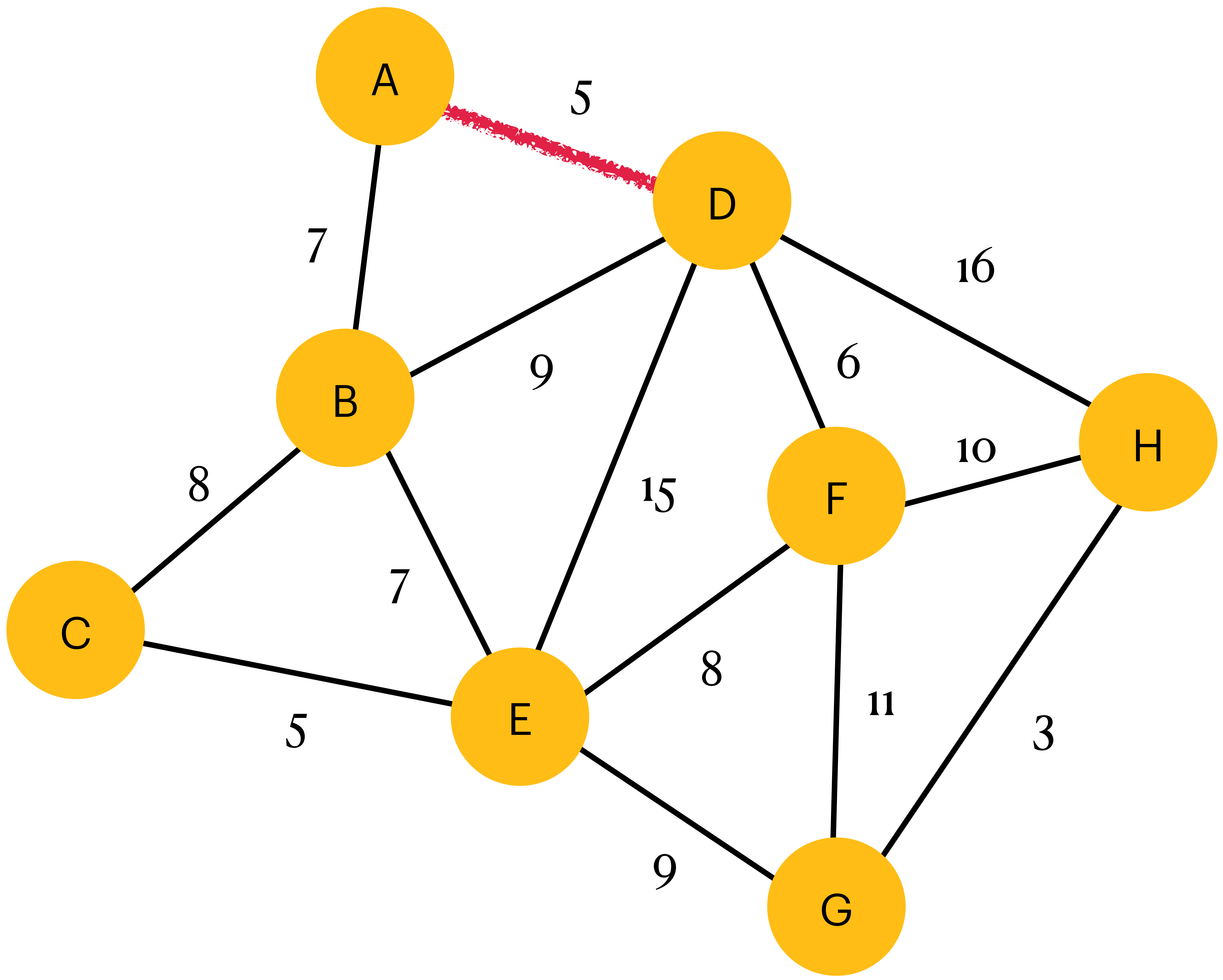
Prim's Algorithm

Example



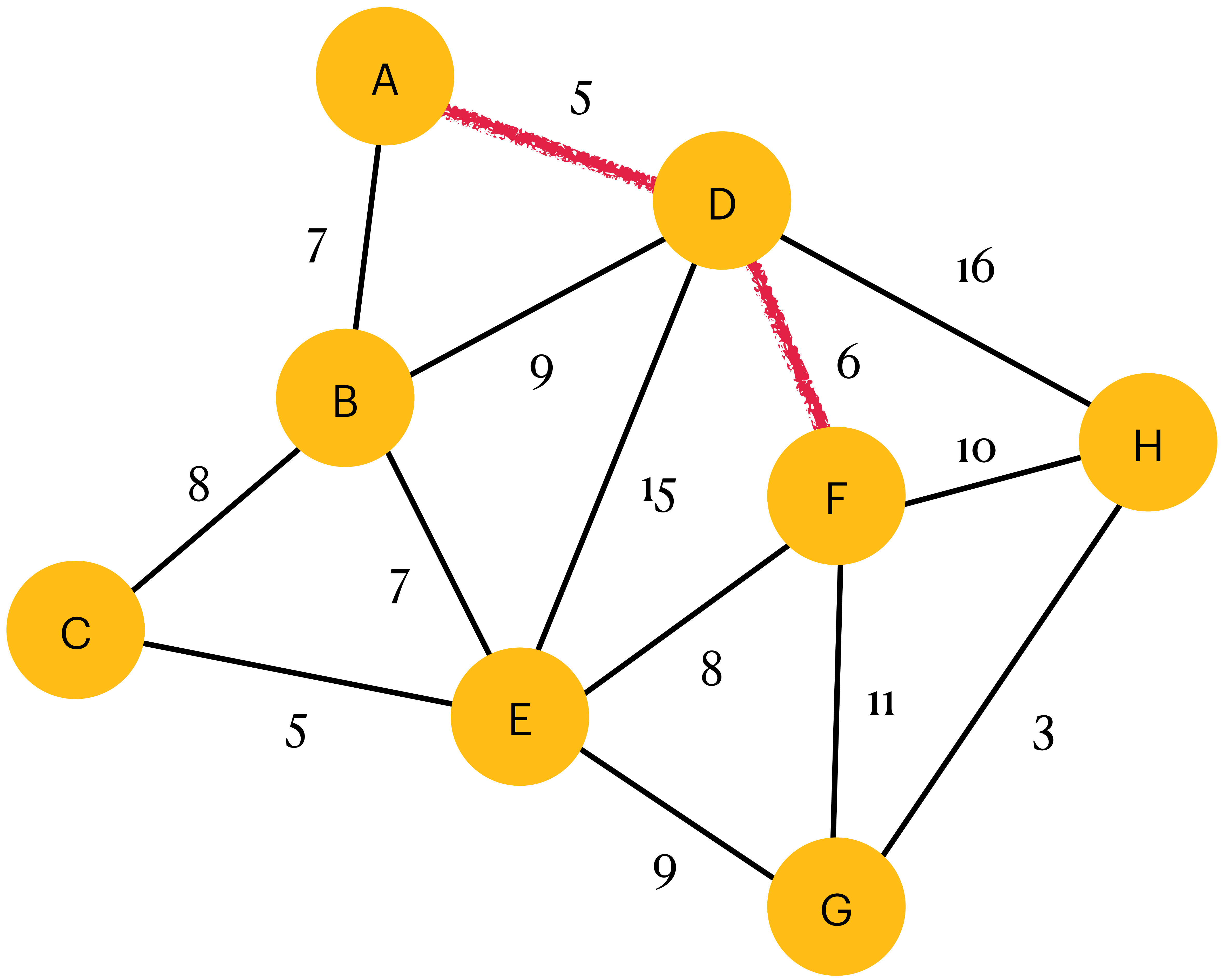
Prim's Algorithm

Example



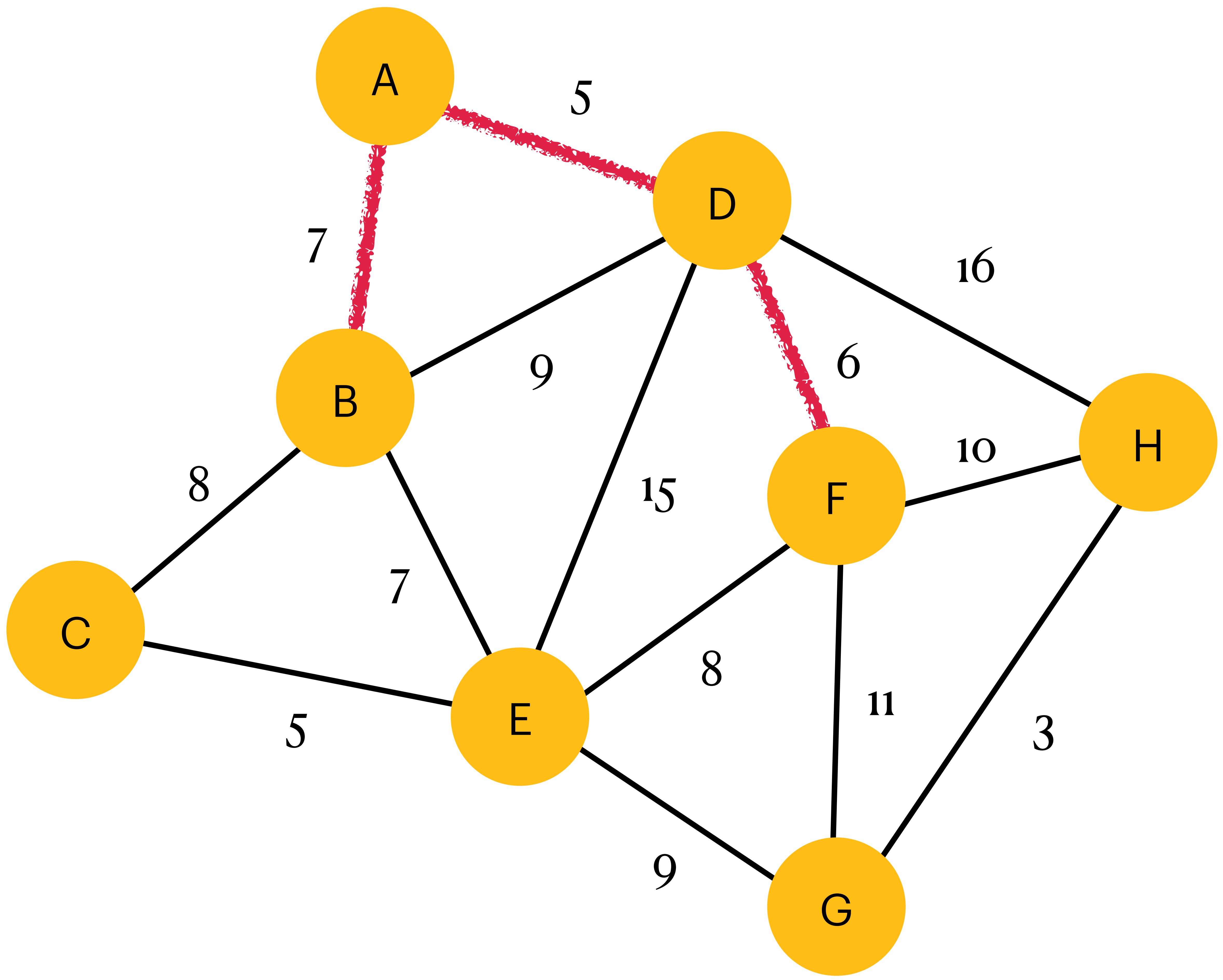
Prim’s Algorithm

Example



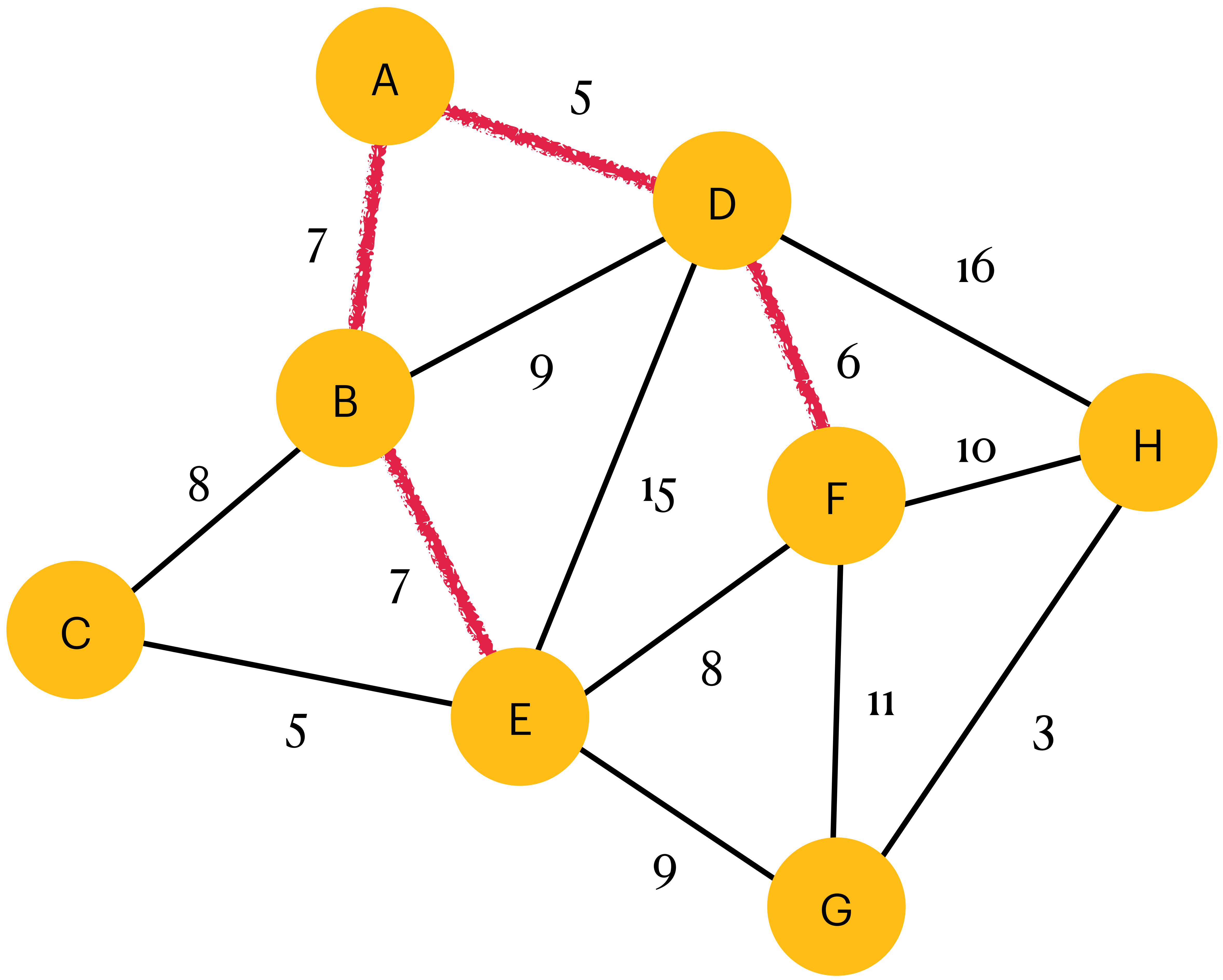
Prim's Algorithm

Example



Prim's Algorithm

Example



Prim's Algorithm

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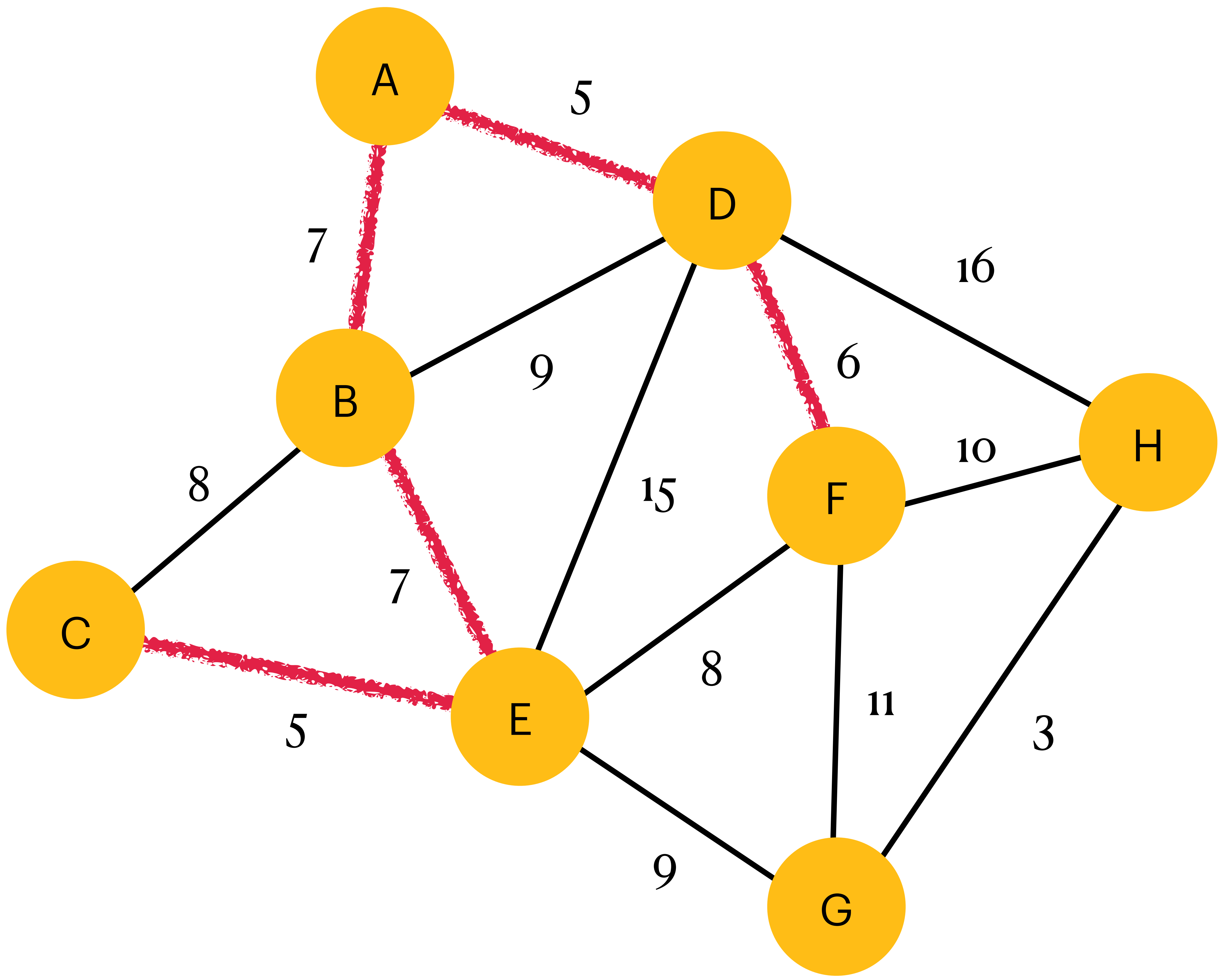
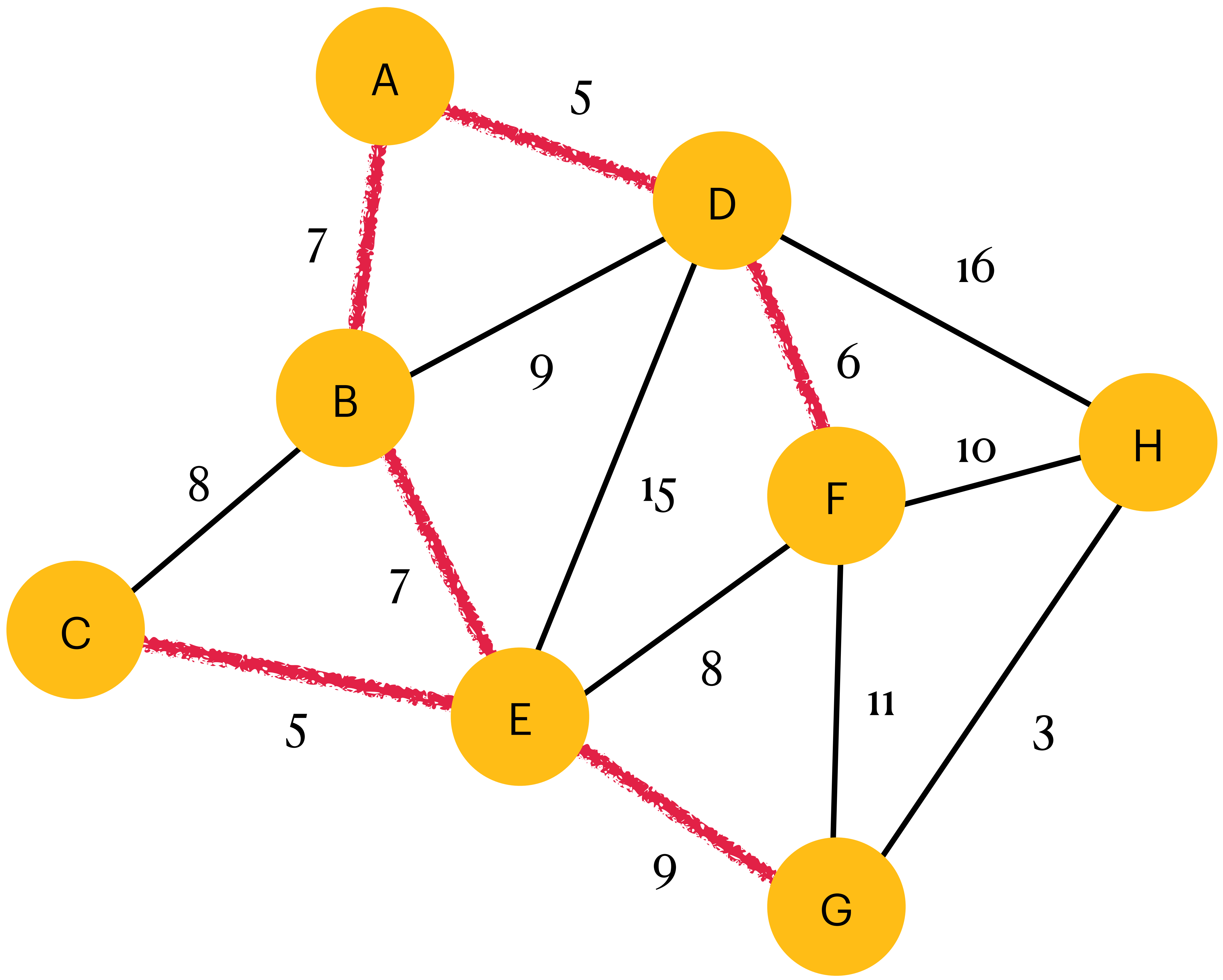


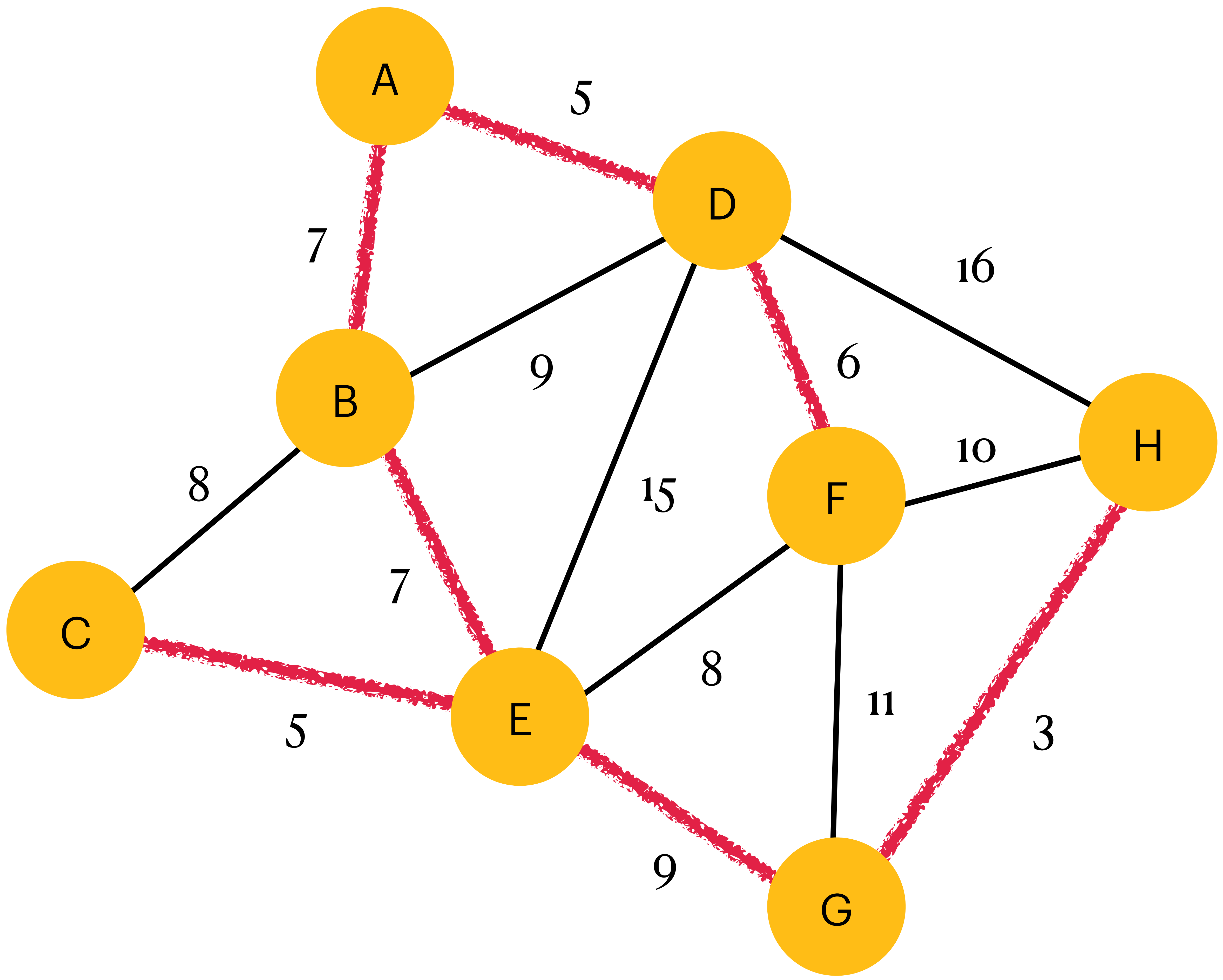
Prim's Algorithm

Example



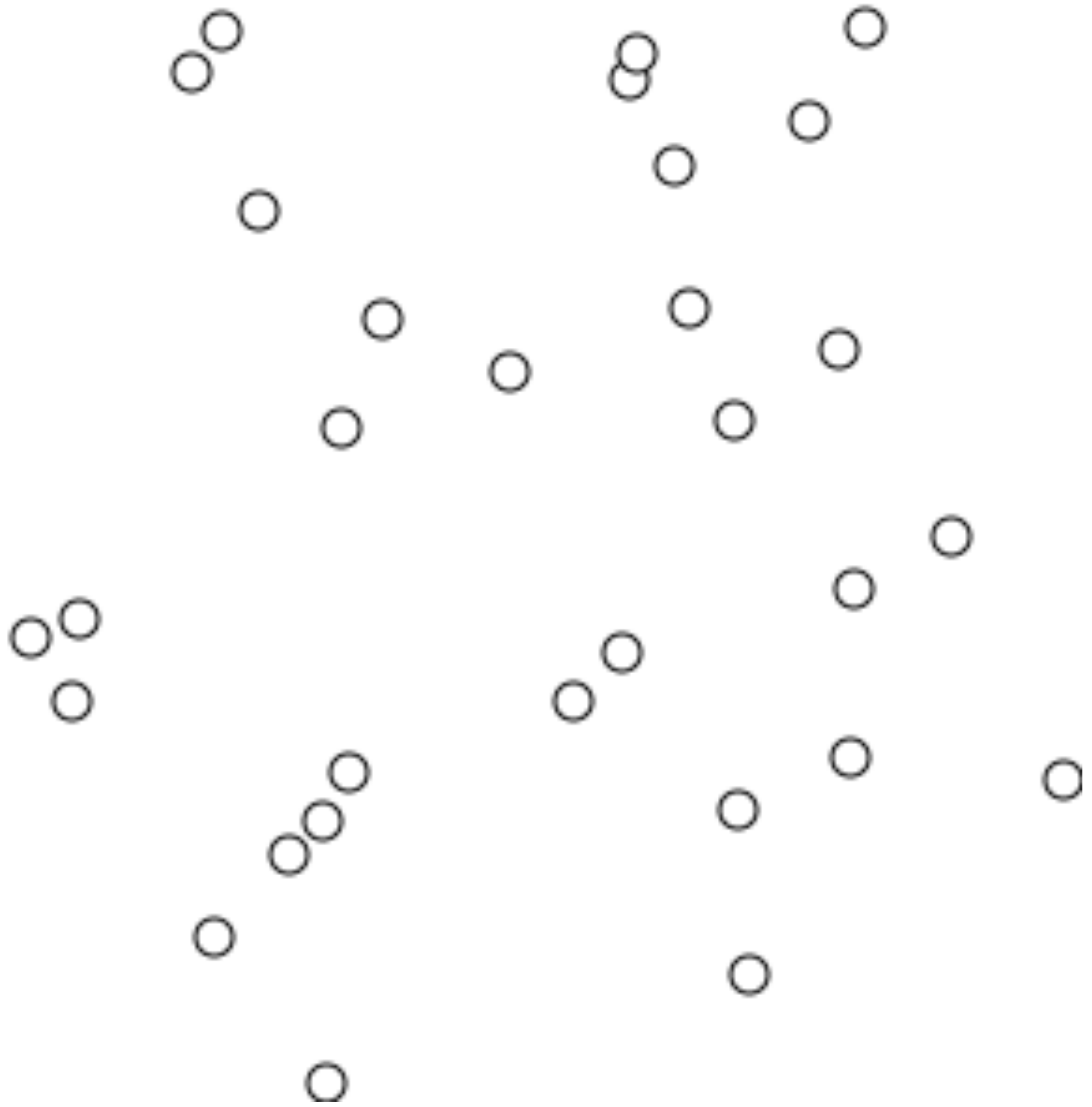
Prim's Algorithm

Example



Kruskal's Algorithm

Idea: Greedily add to the forest a lowest-weight edge that will not form a cycle.



Kruskal's algorithm applied to points in a Euclidean plane.

Kruskal's Algorithm

Algorithm

Procedure:

1. $T \leftarrow \emptyset$
2. **for** $v \in V$ **do**

    MAKE-SET(v)

3. **for** $e = (u, v)$ in E, sorted by weight in non-decreasing order **do**

    if FIND-SET(u) $\neq$ FIND-SET(v) **then**

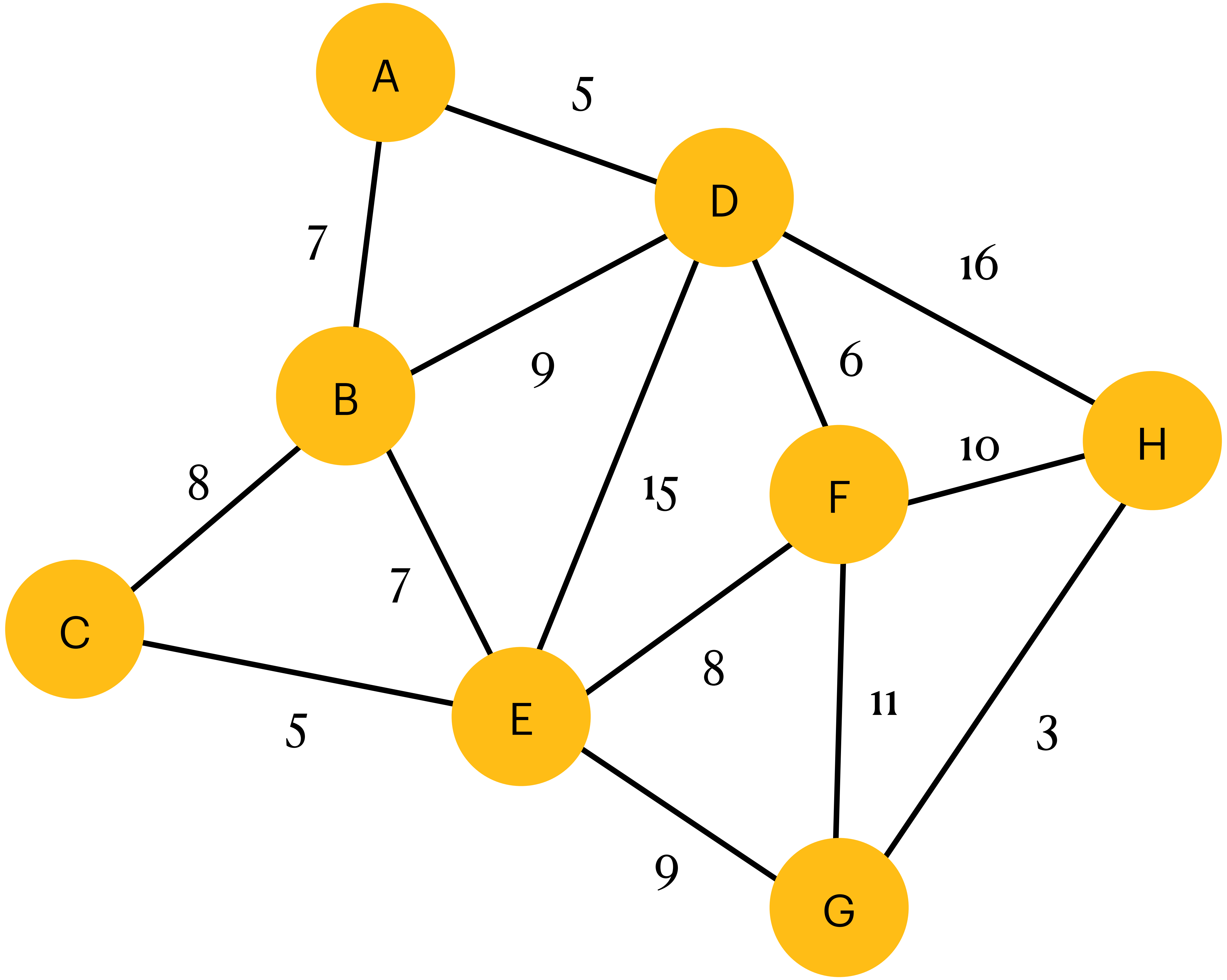
        $T \leftarrow T \cup \{e\}$

        UNION(u, v)

4. **return** T

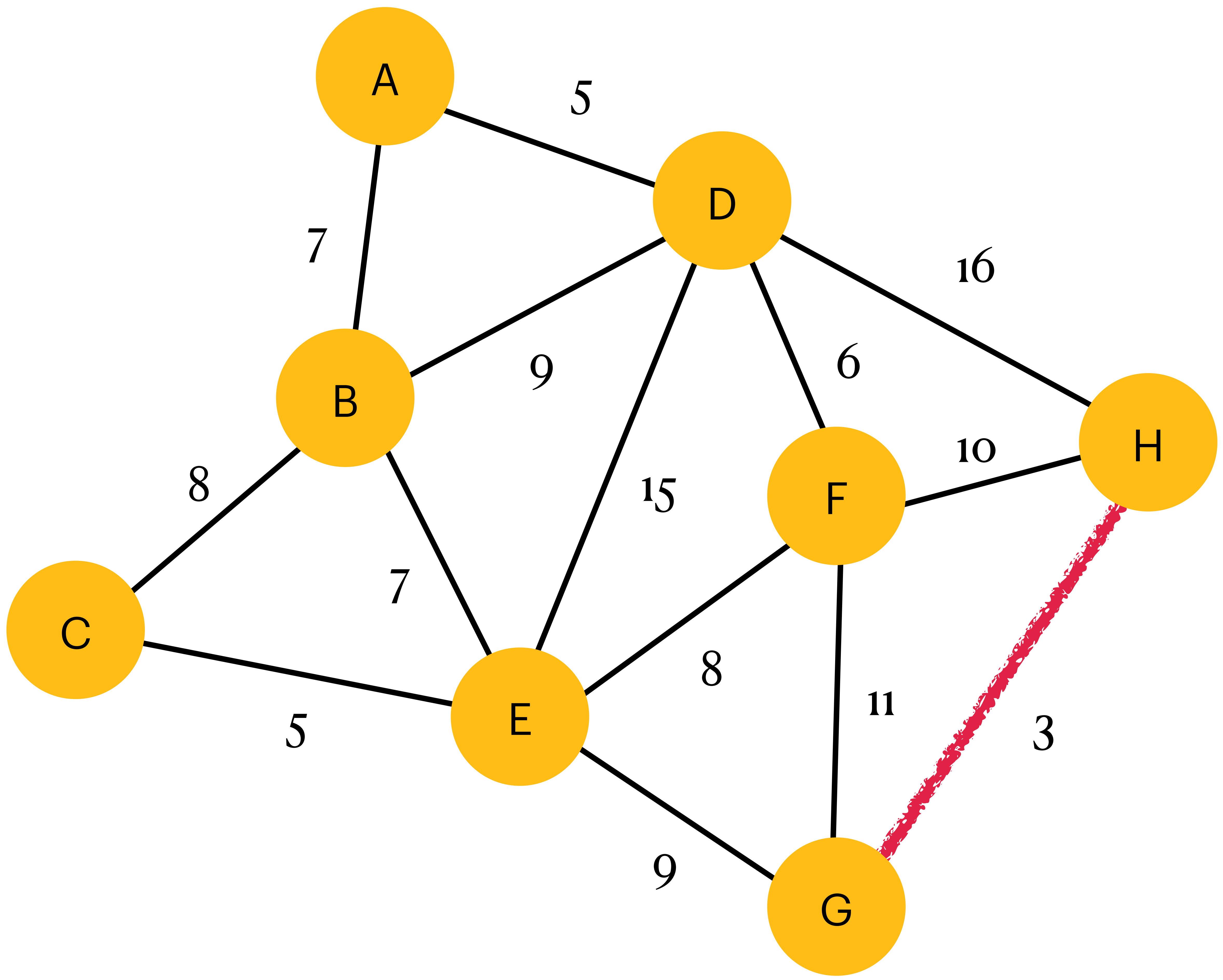
Kruskal’s Algorithm

Example



Kruskal's Algorithm

Example



Kruskal's Algorithm

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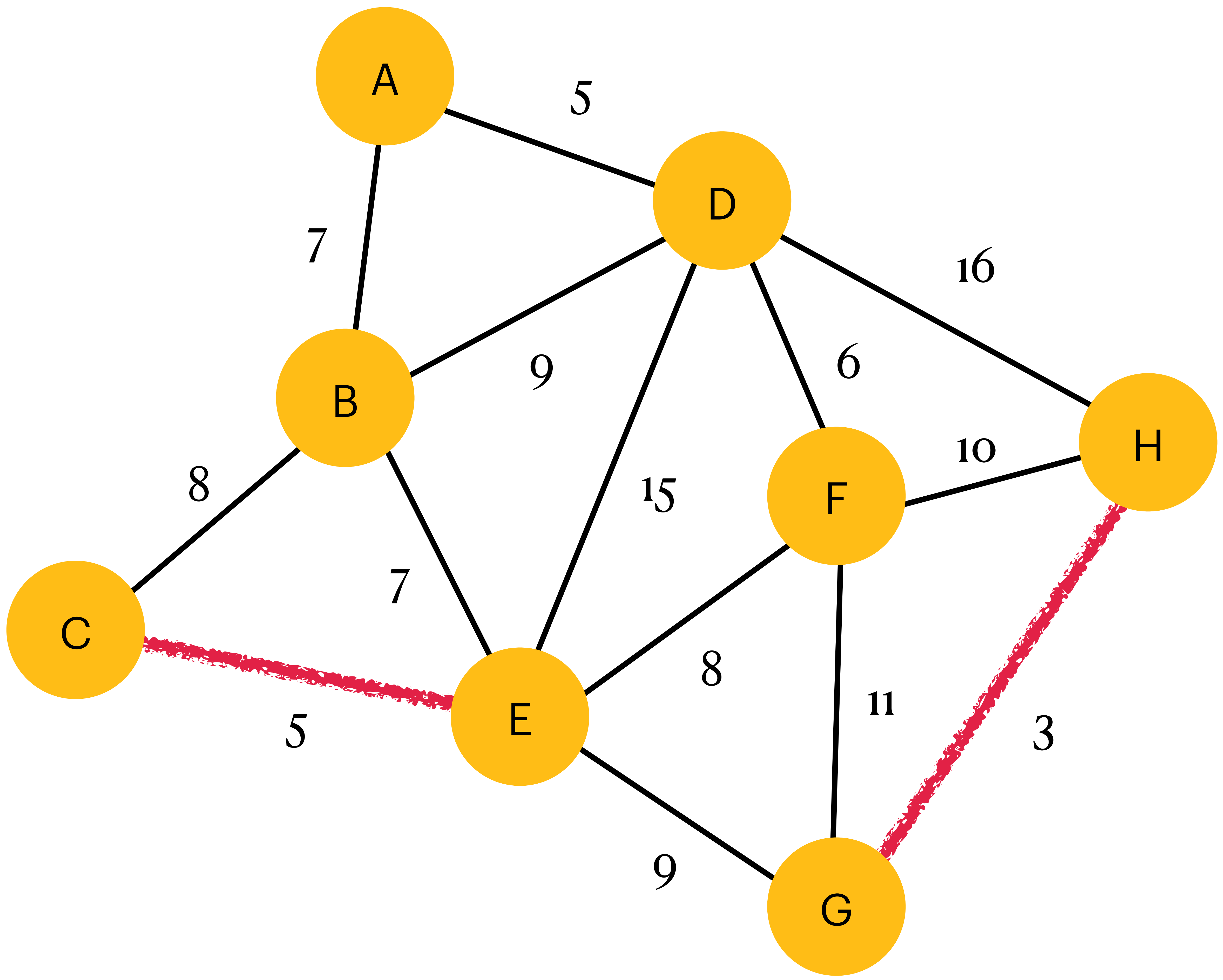

Kruskal's Algorithm

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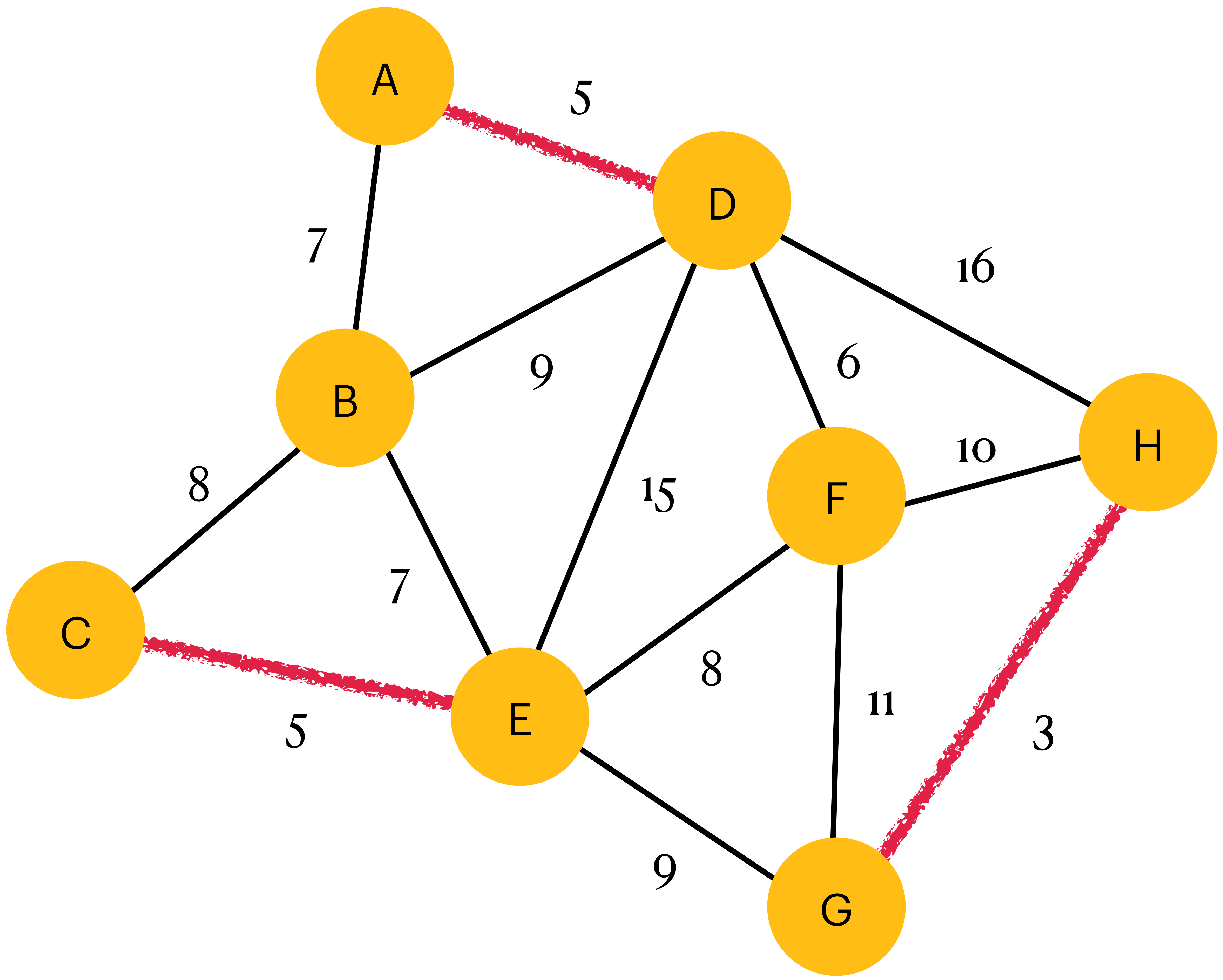

Kruskal's Algorithm

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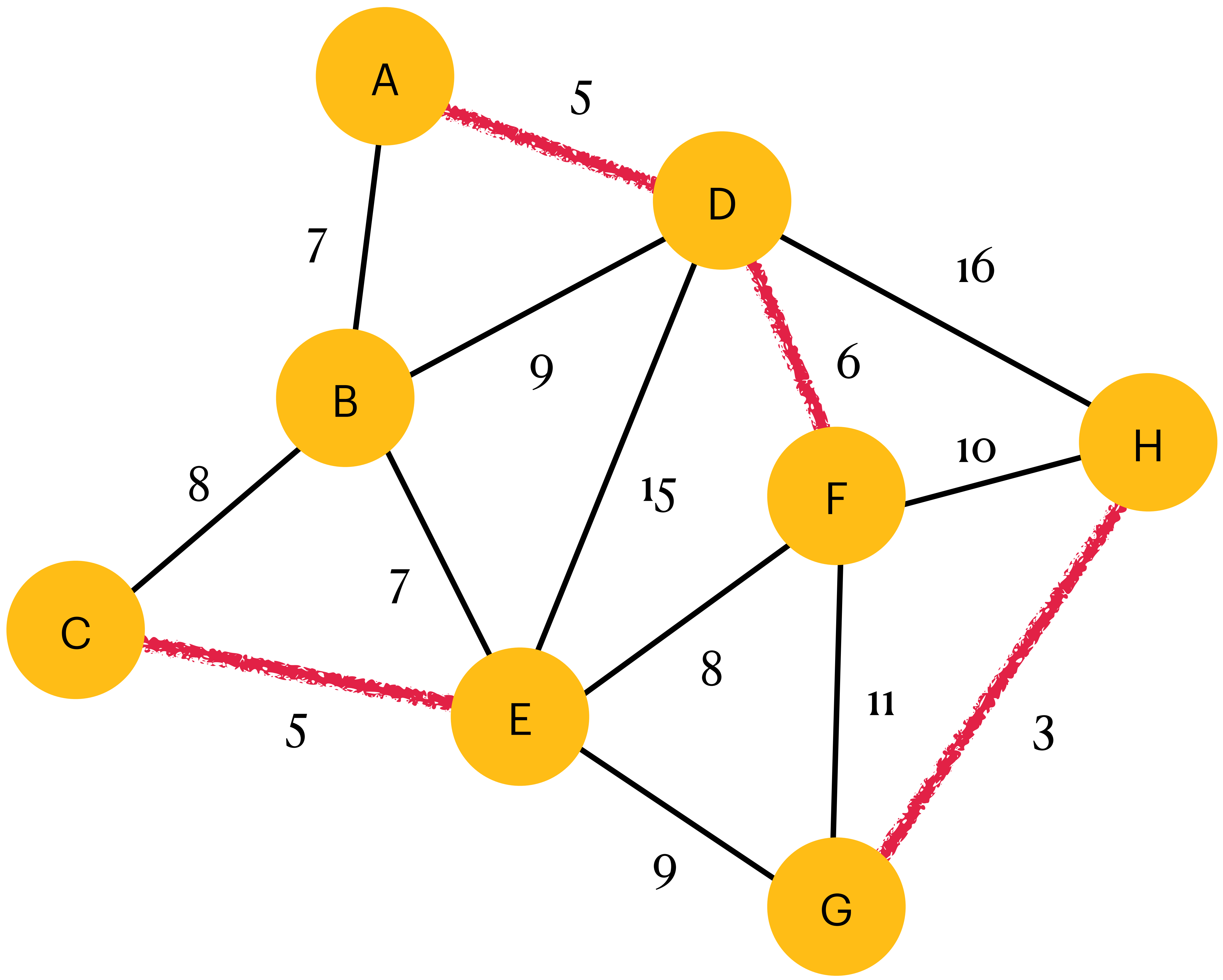

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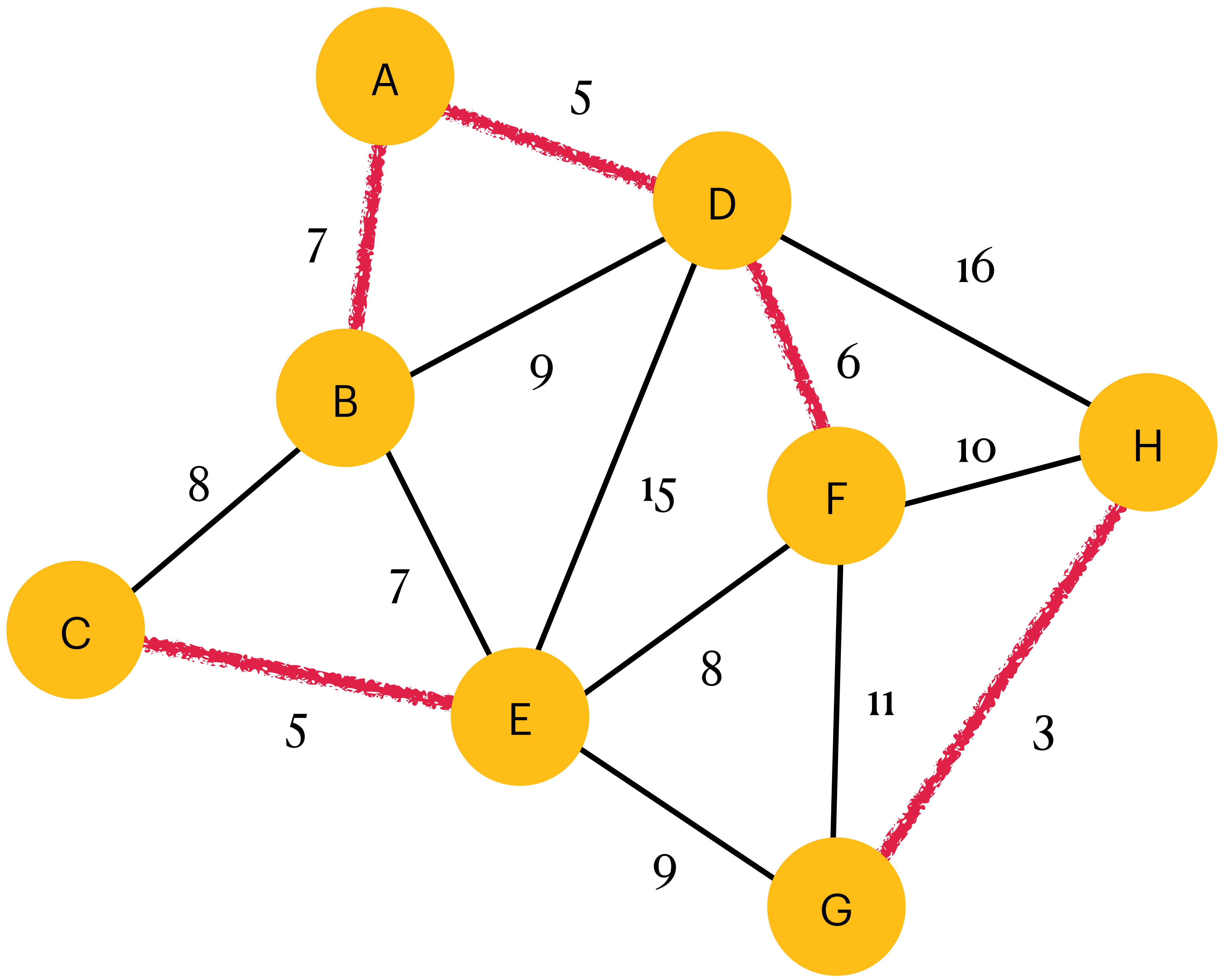

Kruskal's Algorithm

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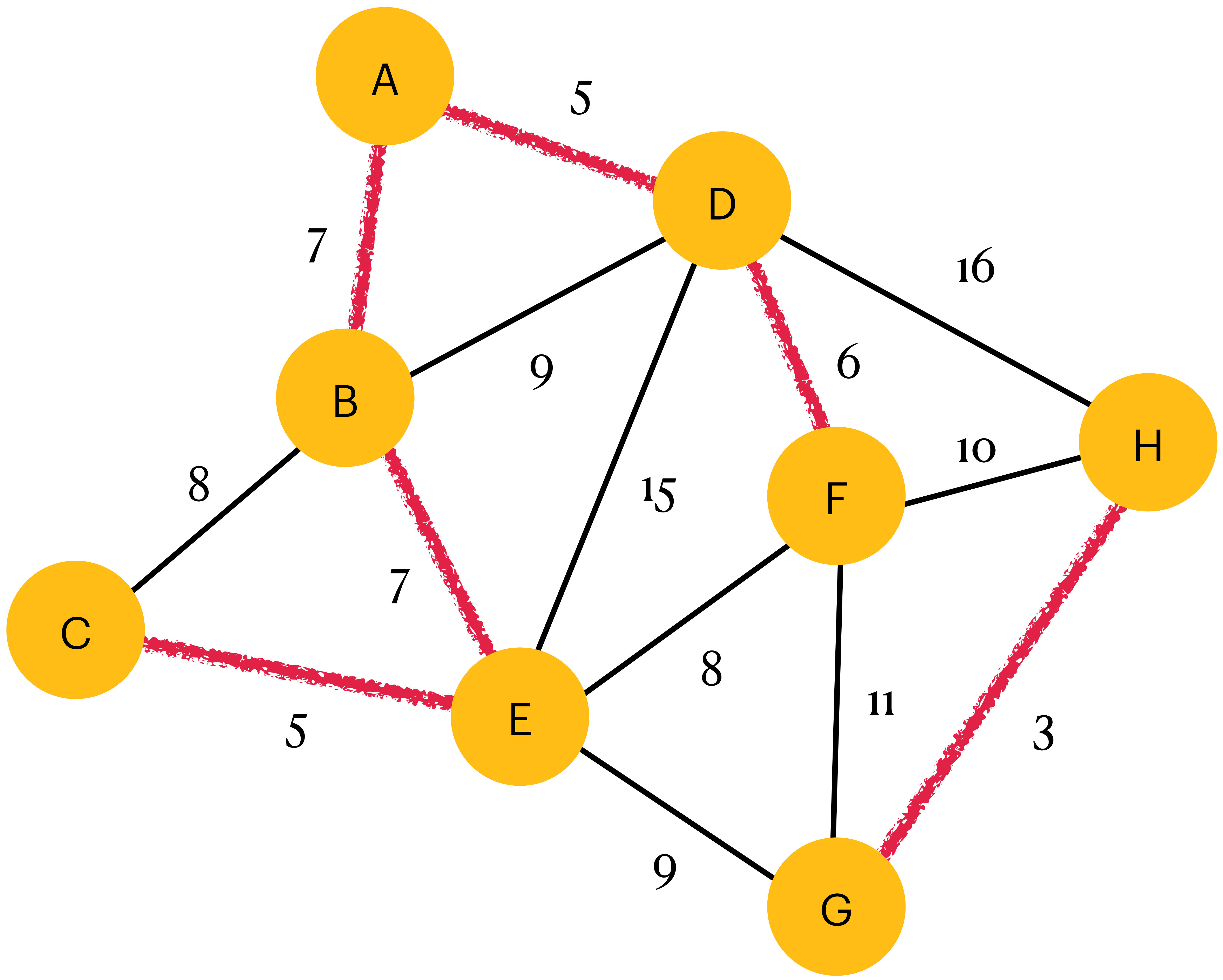
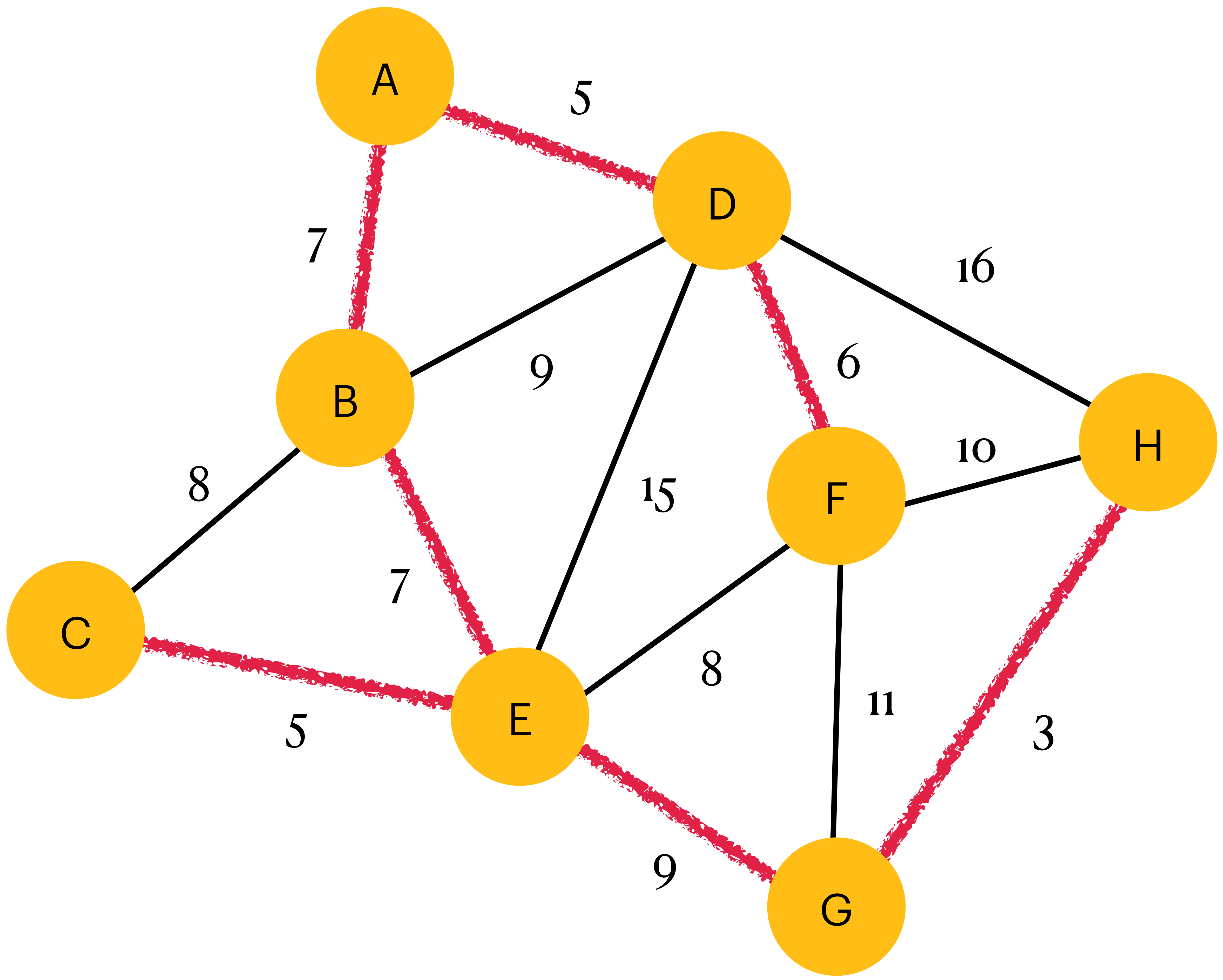


Kruskal's Algorithm

Example



Properties of MSTs

Theorem [Cut Property]: Let S be some non-empty proper subset of the vertices. For the cut $(S, V \backslash S)$, any least-weight edge $e = (u, v)$ with $u \in S, v \notin S$ (i.e., a "crossing" edge) must belong to some MST.

We can greedily construct globally optimal solutions by making locally optimal choices.

Properties of MSTs

Theorem [Cycle Property]: Consider any cycle C in a graph G. If C contains an edge e whose weight is strictly larger than the weight of every other edge in C, then e cannot belong to any MST.

Properties of MSTs

Theorem [Uniqueness Property]: If a graph has distinct edge weights, then its MST is unique.

Practice Problem

What tunnels should I build?

Our subway is described as a graph $G = (V, E, c)$, where the stations are nodes and each possible tunnel between stations is an edge $e \in E$ with unique cost $c(e) > 0$. We find the cheapest way to connect all stations by computing the (unique) MST T of G.

Part A: What if a new edge $e \notin E$ with cost $c(e)$ is now added? Show that we can obtain the MST T' of the modified graph G' by adding e to T and then deleting the heaviest edge in the cycle created.

Proof. Two steps:

1. Only the edges in the set $T'' = T \cup \{e\}$ can be in T'. Why?

2. Removing the heaviest edge in the single cycle in T'' yields the MST. Why?

Part B: Say that the new network is now described by $G' = (V \cup \{v'\}, E \cup E', c)$, where v' is a new station, and E' is the set of possible tunnels between the new station and existing ones. We want to find the MST T' of G' given MST T of G in $O(|V| \log |V|)$ time.