Recitation 6

Maximum Flow

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How’s everyone? How was the quiz?

Overview

Maximum Flow Review

Cuts in Flow Networks

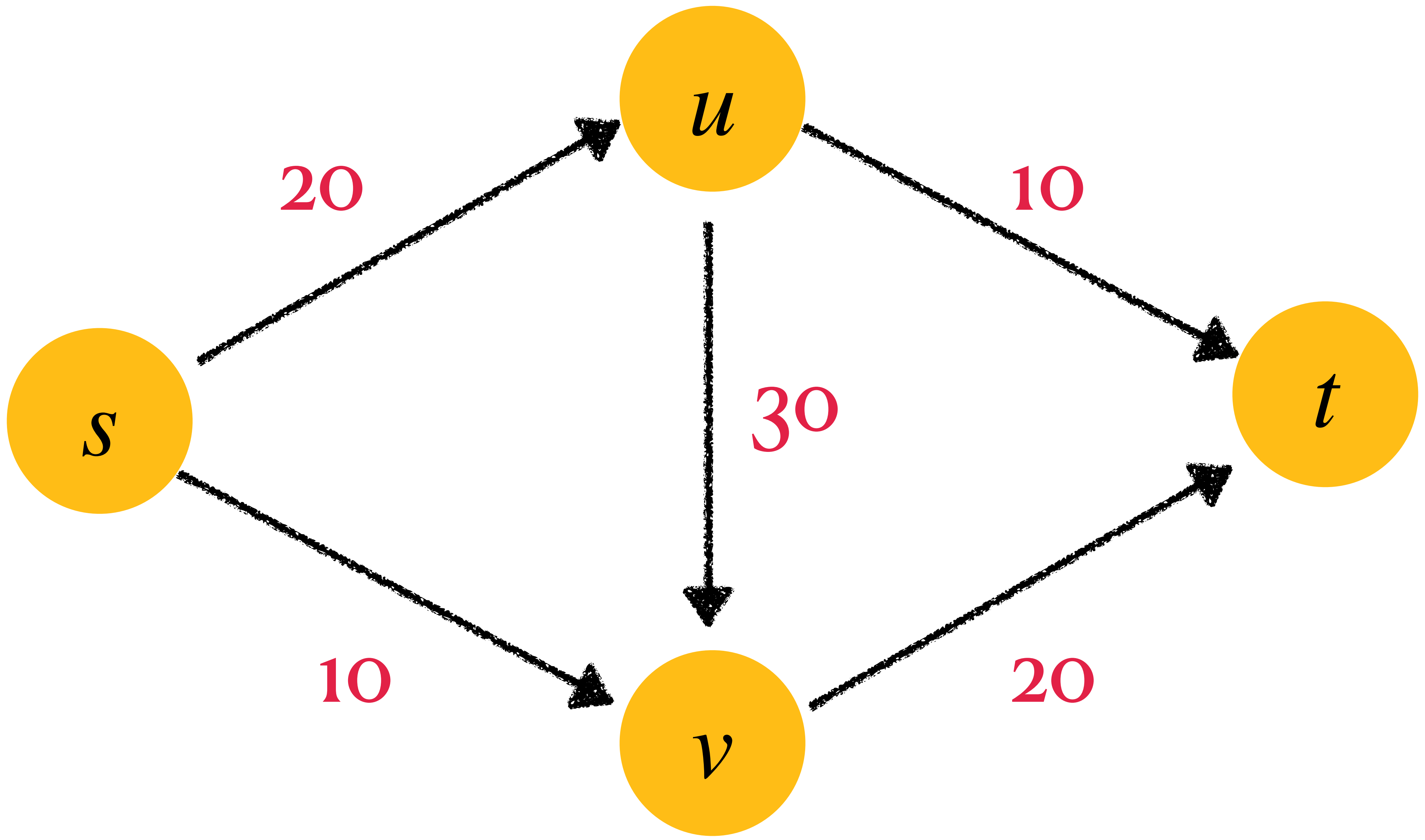
Ford-Fulkerson Algorithm

Practice Problem

Review

Flow Network

A **flow network** $G = (V, E, s, t, c)$ is a directed graph with a *source* node $s \in V$, *sink* node $t \in V$, and *capacity* $c(u, v)$ per each edge $(u, v) \in E$, where $c : E \mapsto \mathbb{R}^{\geq 0}$.



Review

What is the total flow along an edge?

Given a flow network, a **gross flow** is a function $g : E \mapsto \mathbb{R}^{\geq 0}$ satisfying:

1. Capacity (or feasibility) constraints:

   "cannot be negative, cannot exceed capacity"

   For every edge $(u, v) \in E, 0 \leq g(u, v) \leq c(u, v)$

2. Flow conservation constraints:

   "flow coming in = flow going out"

   For every node $v \in V \backslash \{s, t\}$, $\sum_{u \in \delta^-(v)} g(u, v) = \sum_{u \in \delta^+(v)} g(v, u)$

Review

What is the total flow between two vertices?

A **net flow** is a function $f : V \times V \mapsto \mathbb{R}^{\geq 0}$ satisfying:

1. Capacity (feasibility) constraints:

   For all u and v, $f(u, v) \leq c(u, v)$

2. Flow conservation constraints:

   For all $u \in V \backslash \{s, t\}$, $\sum_{v \in V} f(u, v) = 0$

3. Skew symmetric:

   For all u and v, $f(u, v) = -f(v, u)$

The **value** of a flow f is $\sum_{v \in V} f(s, v)$, denoted $|f|$.

Maximum Flow Problem:
Given a flow network $G = (V, E, s, t, c)$, fined a flow with maximum value.

Implicit Summation Notation

Definitions

Notation: For $X, Y \subseteq V$:

Let $f(u, X) = \sum_{v \in X} f(u, v)$ "the total flow from node u to set X"

Ex: Flow conversation: $f(u, V) = 0$ for all $u \in V \backslash \{s, t\}$

Ex: $|f| = f(s, V)$

Let $f(X, Y) = \sum_{x \in X} \sum_{y \in Y} f(x, y)$ "the total flow from the set X to the set Y"

Implicit Summation Notation

Properties

Properties:

1. For every $X \subseteq V, f(X, X) = 0$
2. For every $X, Y \subseteq V, f(X, Y) = -f(Y, X)$
3. For every $X, Y, Z \subseteq V$, with X and Y disjoint, $f(X \cup Y, Z) = f(X, Z) + f(Y, Z)$

Implicit Summation Notation

Exercise

Claim: $|f| = f(V, t)$

Proof.

$$
\begin{aligned}
|f| &= f(s, V) \\
&= f(V, V) - f(V \setminus \{s\}, V) \quad \text{(property 3)} \\
&= f(V, V \setminus \{s\}) \quad \text{(property 1 \& 2)} \\
&= f(V, t) + f(V, V \setminus \{s, t\}) \quad \text{(property 3)} \\
&= f(V, t)
\end{aligned}
$$

Cuts in Flow Networks

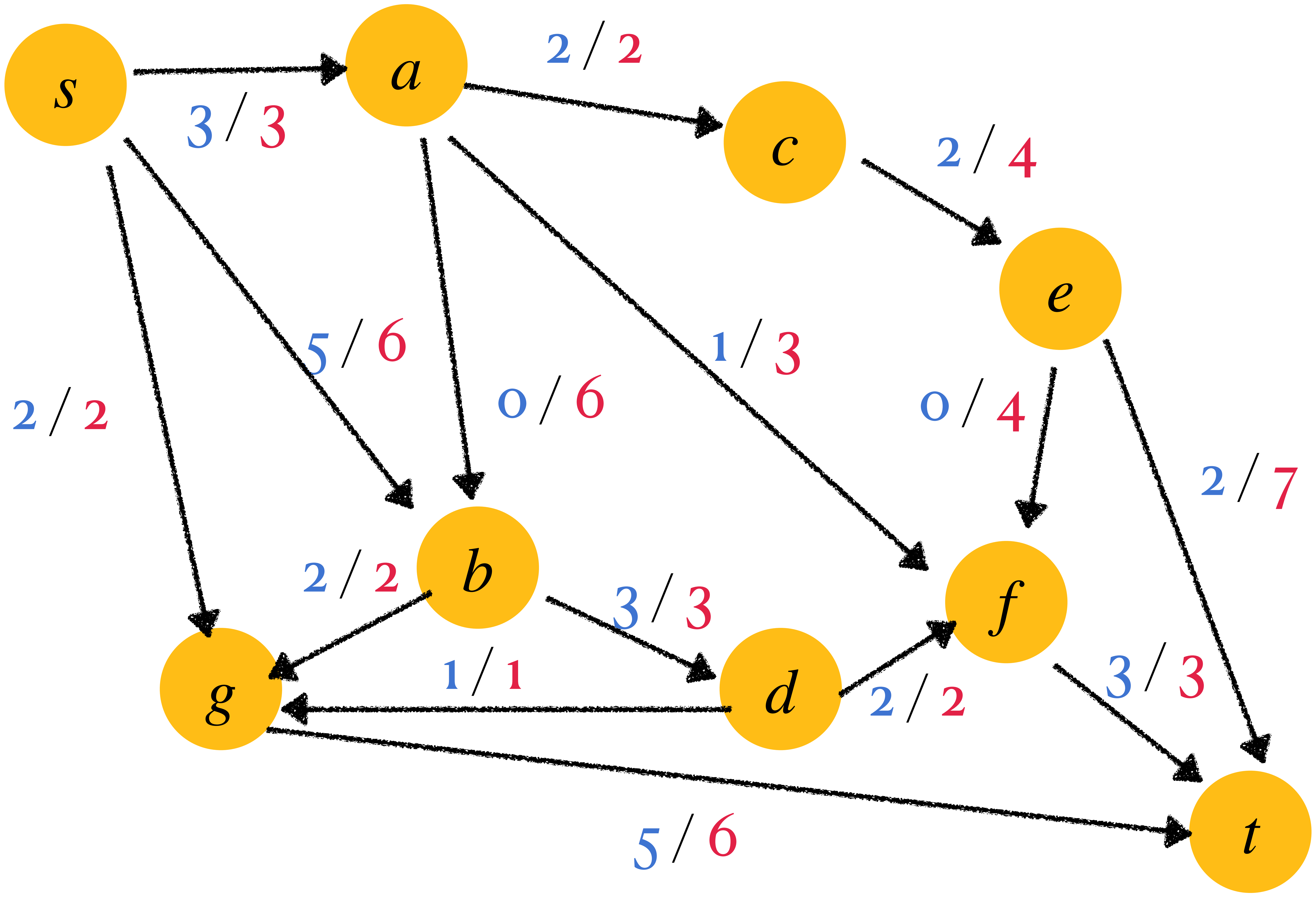
Definitions

An ***s*-*t* cut** of a flow network G is a partitioning of its vertices V into two non-empty subsets S and $T = V \backslash S$ such that S contains the source s and T contains the target t.

- Given a flow f, the net flow $f(S)$ across the cut from S to T is $f(S, T)$
- The capacity of the cut is $c(S) = c(S, V \backslash S)$

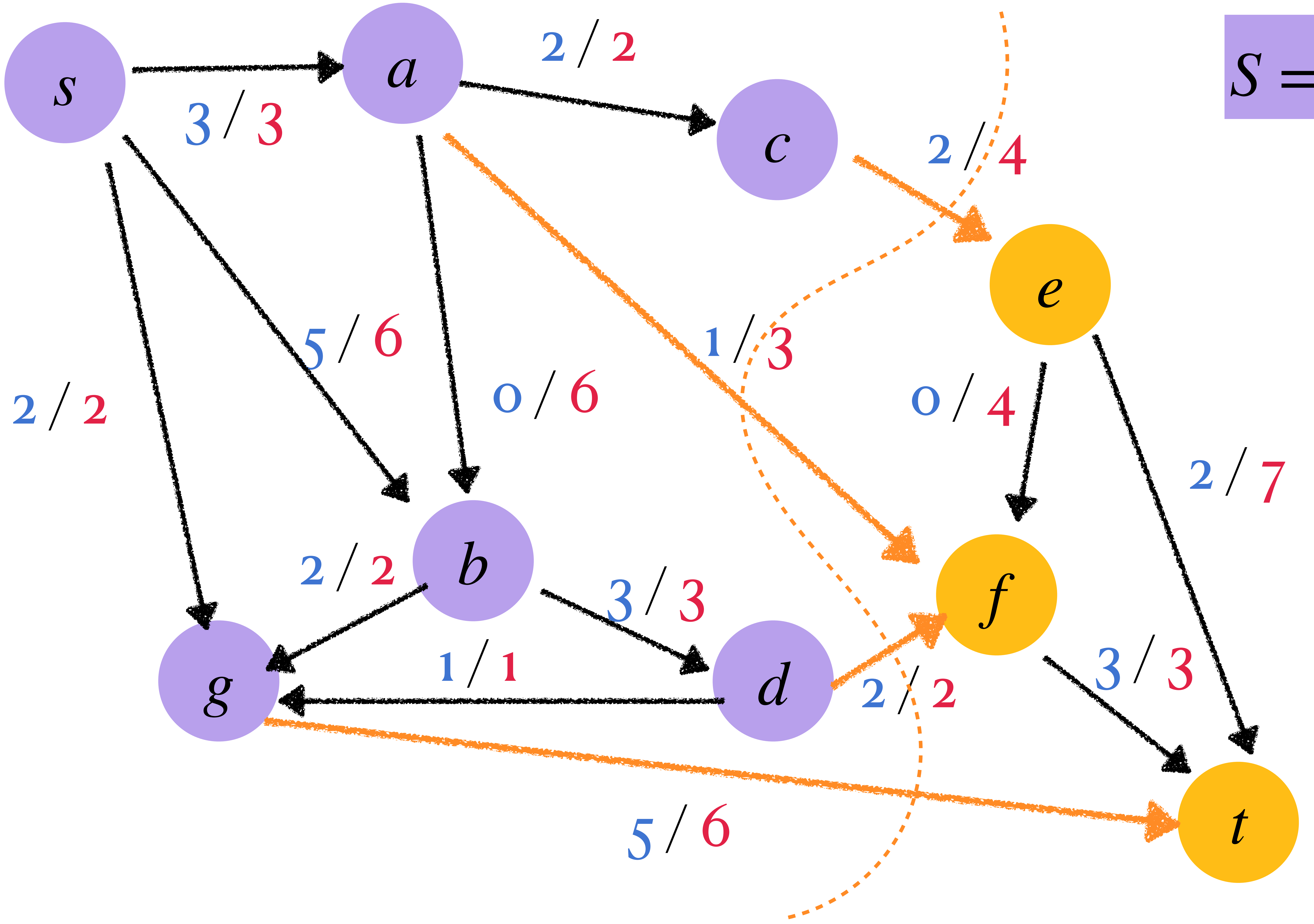
Cuts in Flow Networks

Example



Cuts in Flow Networks

Example



$S = \{s, a, b, c, d, g\}$

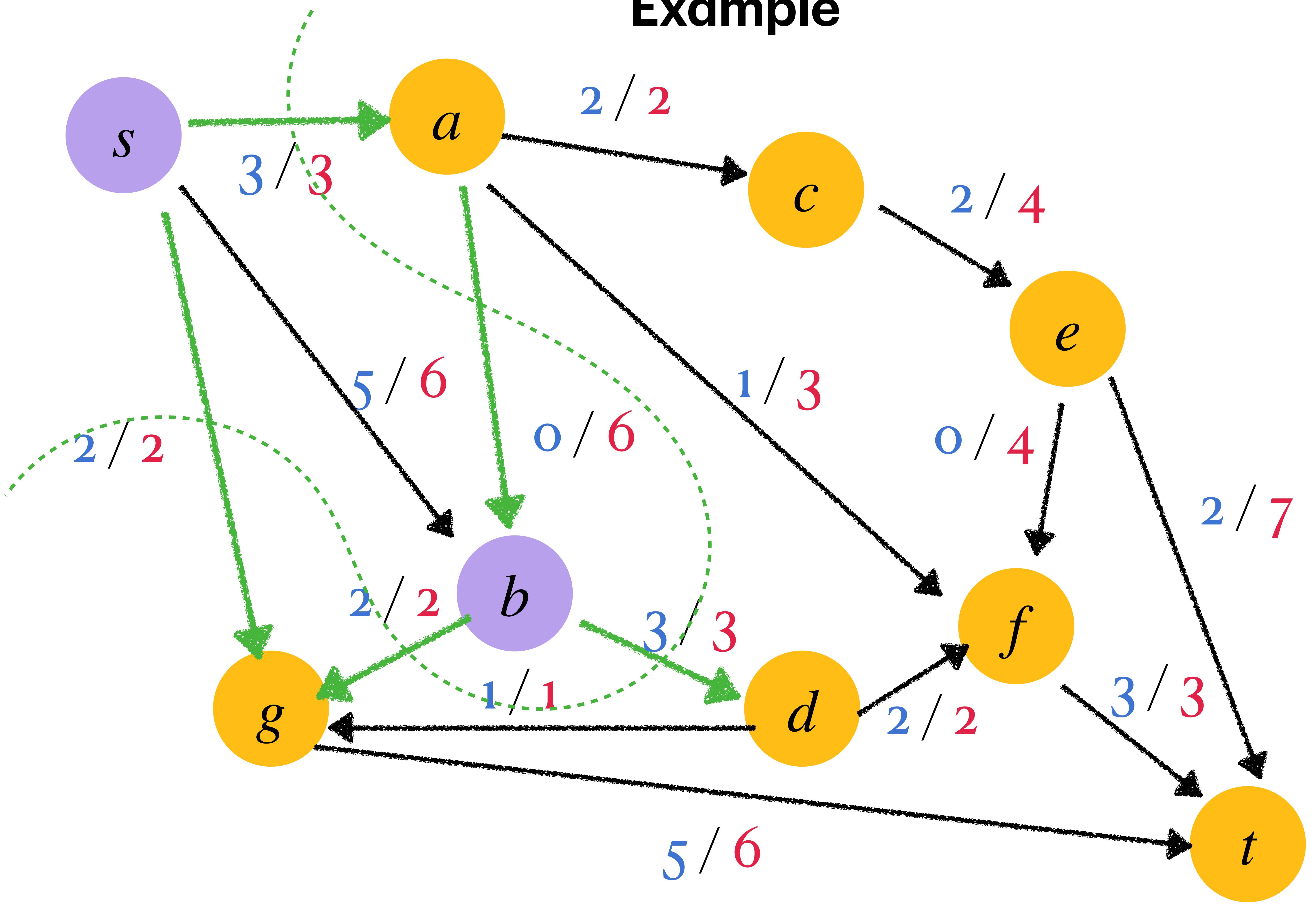
$c(S) = 15$

$f(S) = 10$

Cuts in Flow Networks

Example

$S = \{s, b\}$

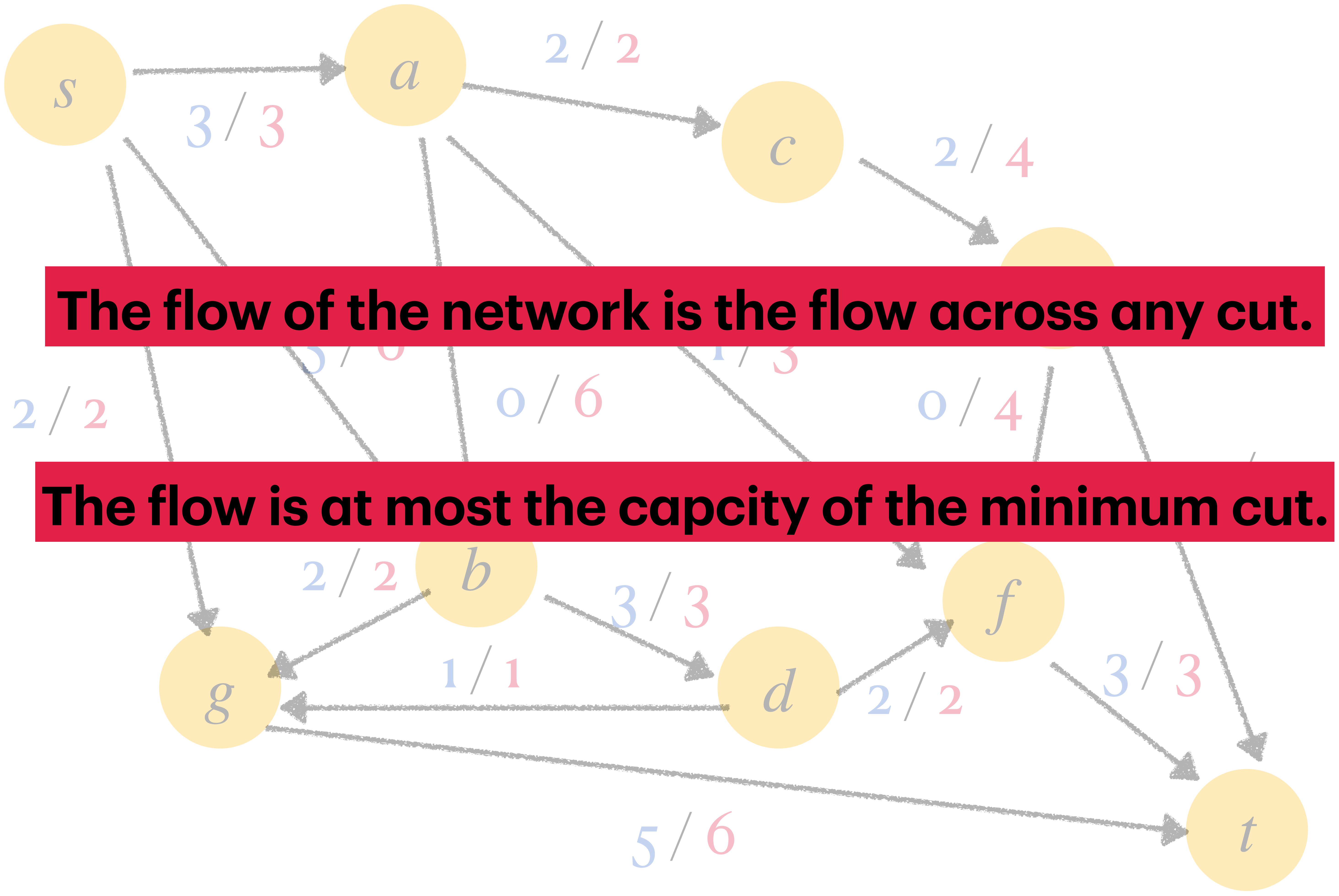


$c(S) = 10$

$f(S) = 10$

Cuts in Flow Networks

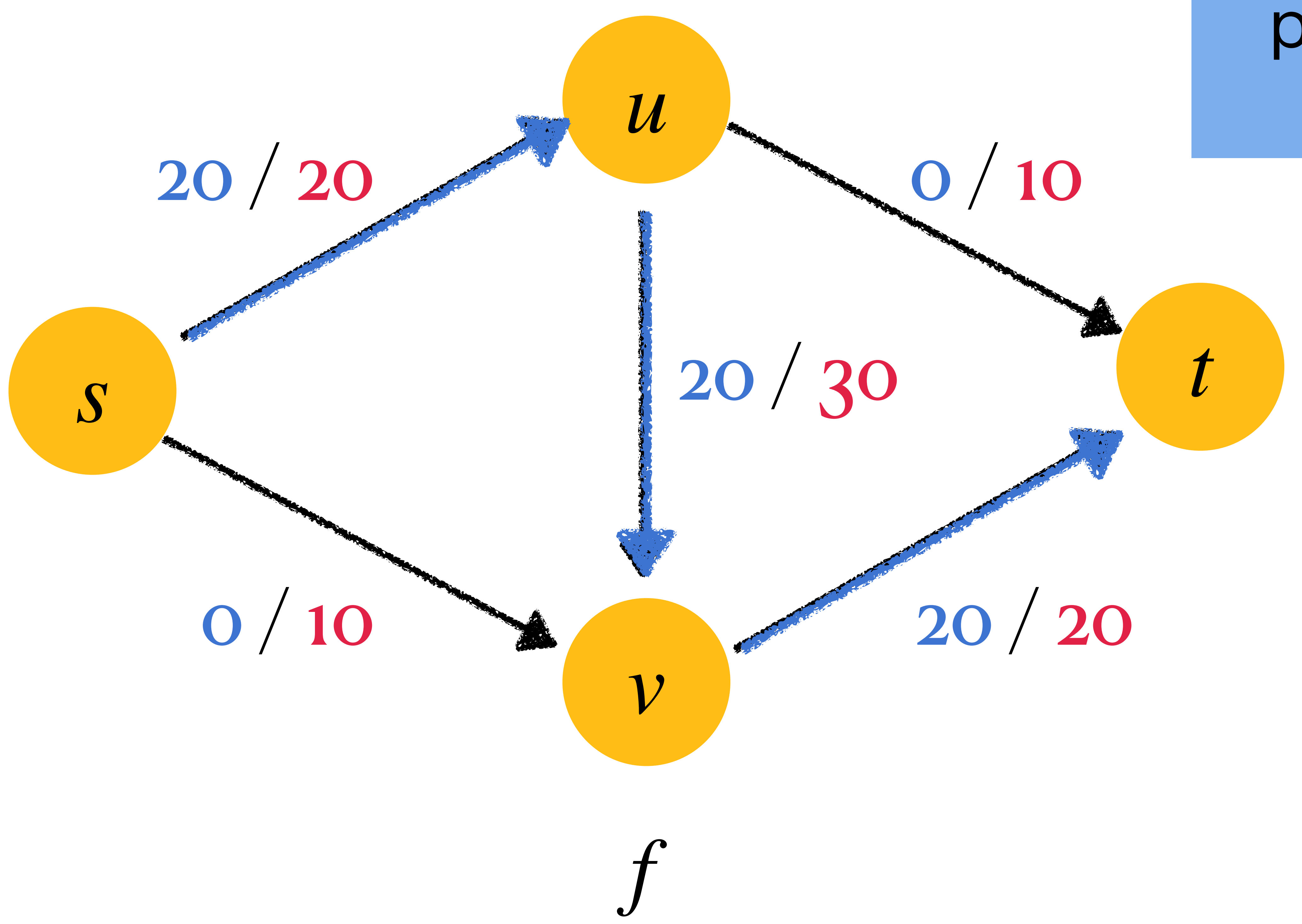
Example



The flow of the network is the flow across any cut.

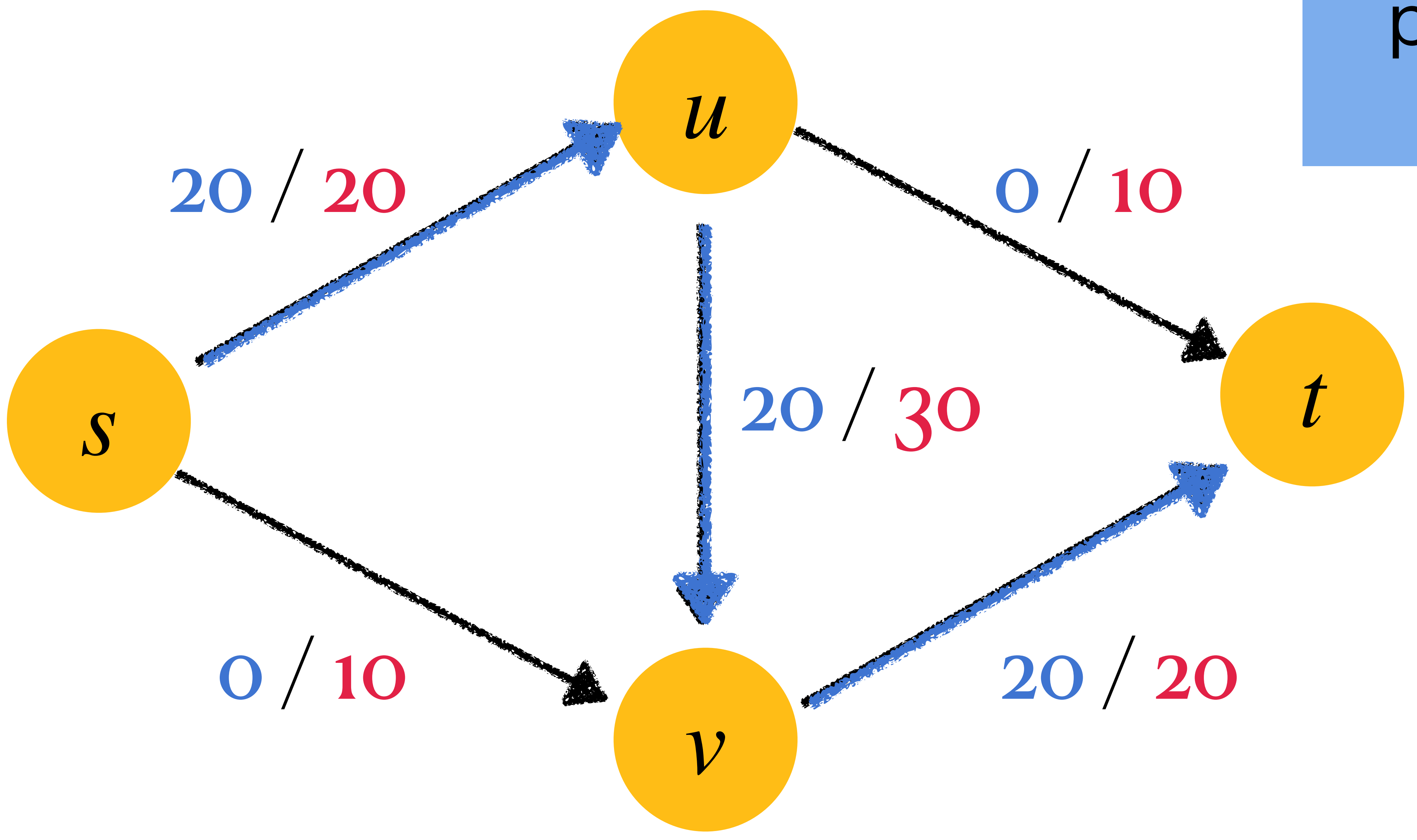
The flow is at most the capcity of the minimum cut.

Residual Networks and Augmenting Paths



Where can we push more flow?

Residual Networks and Augmenting Paths



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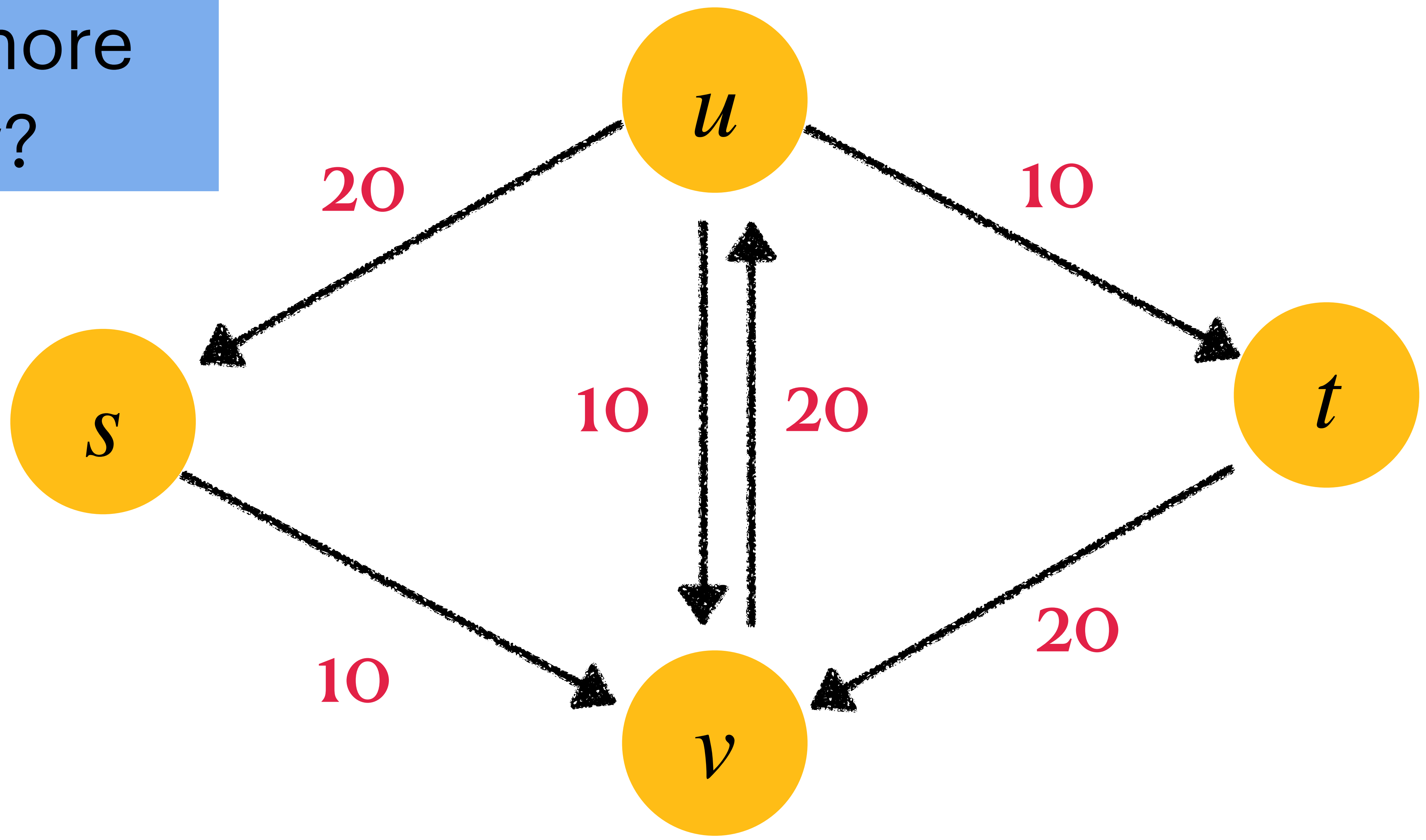
Where can we push more flow?

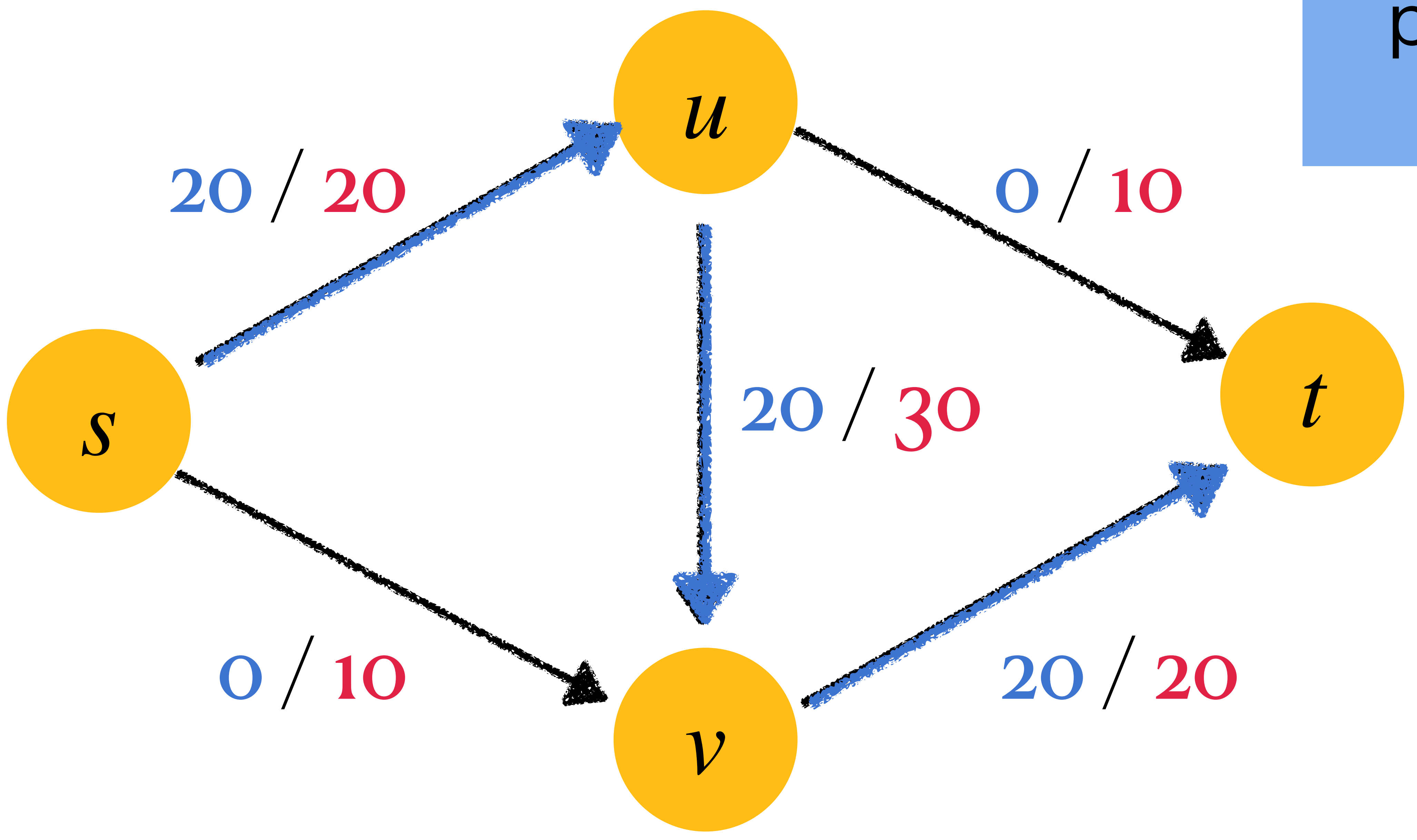
Residual Graph G_f



residual capacity:

$c_f(u, v) = c(u, v) - f(u, v)$

Residual Networks and Augmenting Paths



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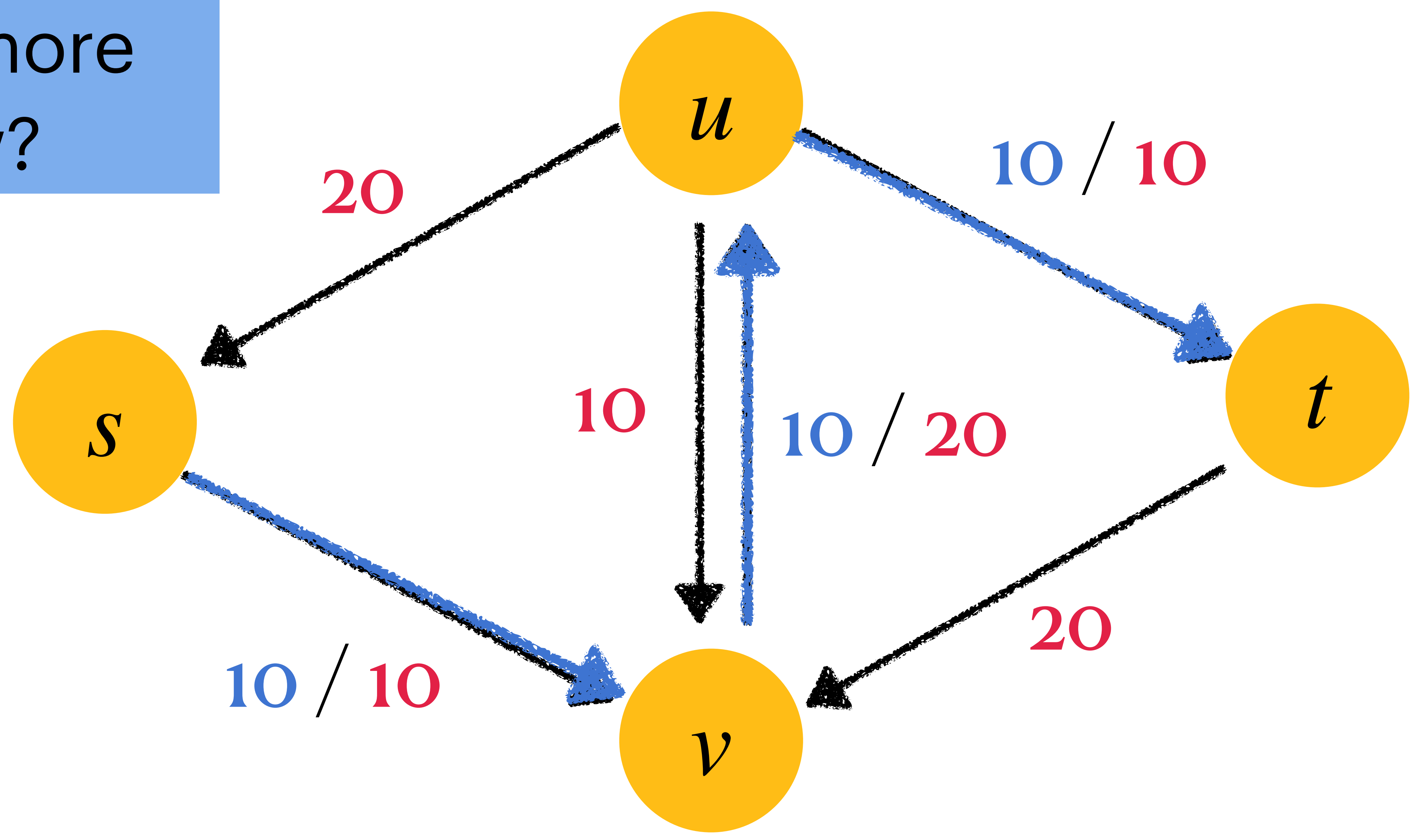
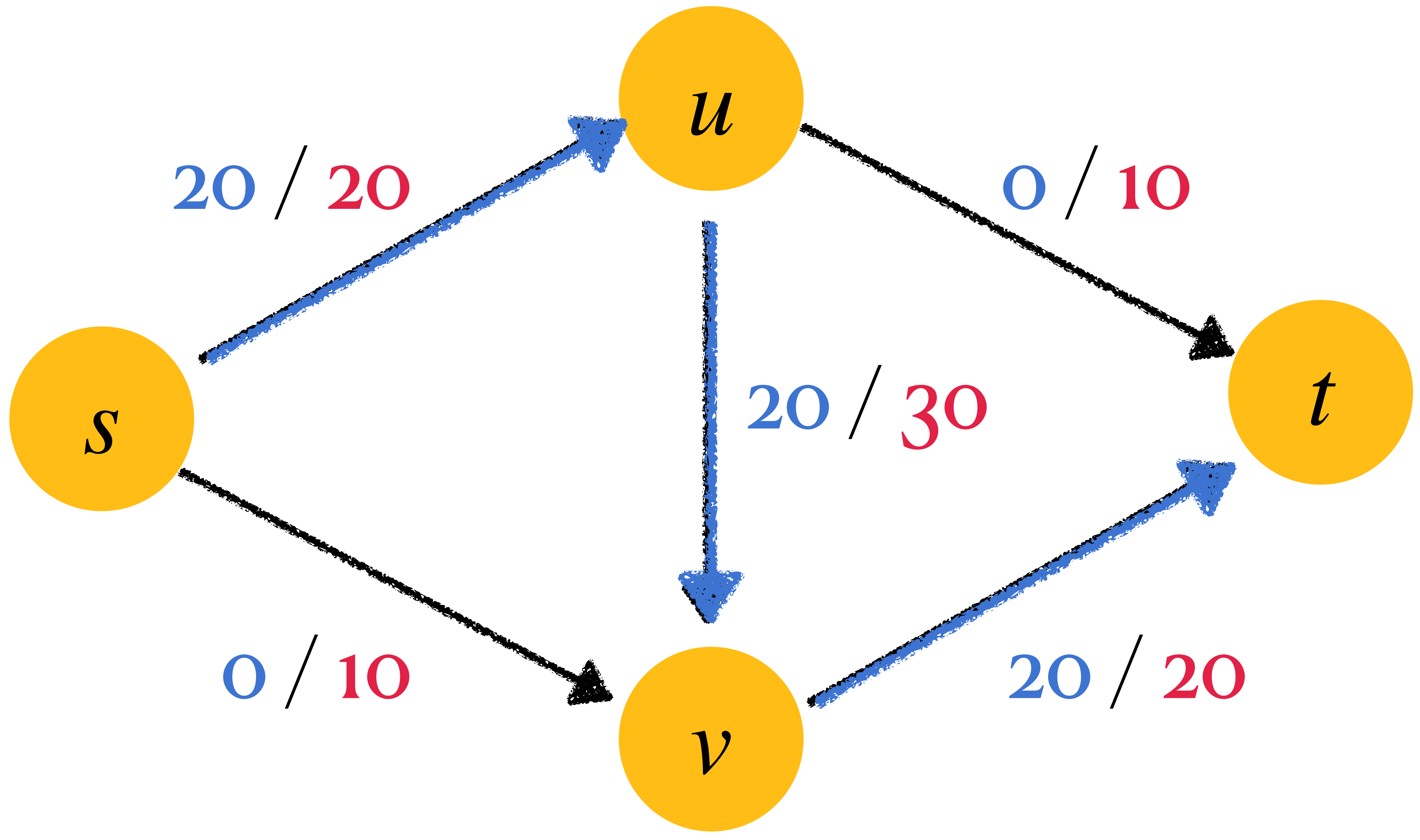
Where can we push more flow?

Residual Graph G_f



We can push flow through this augmenting path, with bottleneck capcity 10.

Residual Networks and Augmenting Paths



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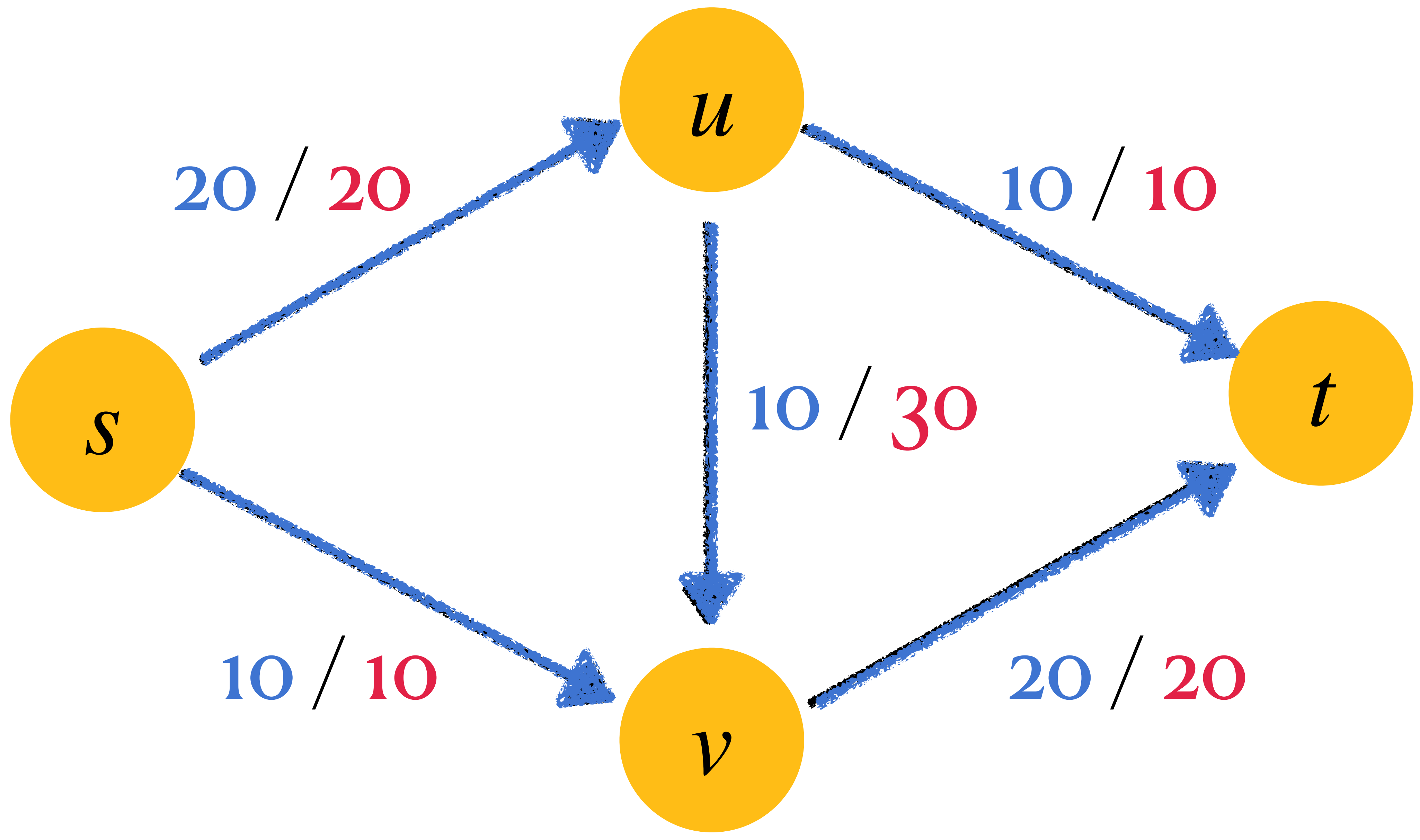

f'

Max-Flow Min-Cut

Max-Flow Min-Cut Theorem: Suppose f is a flow in a flow network G. The following are equivalent:

1. $|f| = c(S)$ for some s-t cut $(S, V \backslash S)$
2. f is a maximum flow
3. f admits no augmenting paths

Ford-Fulkerson

Given $G = (V, E, s, t, c)$:

1. $f \leftarrow 0$
2. **while** there exists an augomenting path P in G_f **do**

   Augment flow f along P with its bottleneck capacity $c_f(P) = \min_{e \in P} c_f(e)$

3. **return** f

Flow Integrality Theorem

Flow Integrality Theorem: If the capacities of all edges are integers, then the value of the maximum flow is an integer as well. Fruthermore, there exists an integral maximum flow f such that $f(u, v)$ is an integer for any $u, v \in V$.

3.2 Edmonds-Karp

```
EDMONDS-KARP(G = (V, E, s, t, c))
Initialize flow f to 0
while there is an augmenting path in G_f do
    Find such a path P using BFS
    Augment flow f along P with its bottleneck capacity c_f(P) = min_{e∈P} c_f(e)
return f
```