Recitation 8

Linear Programming II & Game Theory

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Recall

Duality Theorems

Weak Duality:

The objective value corresponding to any feasible solution of the primal maximization LP is *upper bounded* by the objective value corresponding to any feasible solution of the dual minimization LP

Strong Duality:

The optimal objective value of the primal LP *equals* the optimal value of the dual LP if either one of them is bounded

Theorem: If the primal and dual LPs are both feasible, then both programs are bounded and strong duality holds.

Worked Example

Primal

$$
\begin{array}{ll}
\text{max} & x + 2y \\
\text{subject to} & y \le 4 \\
 & x \le 5 \\
 & x + y \le 8 \\
 & x, y \ge 0
\end{array}
$$

Dual

$$
\begin{array}{ll}
\text{min} & 4a + 5b + 8c \\
\text{subject to} & b + c \ge 1 \\
 & a + c \ge 2 \\
 & a, b, c \ge 0
\end{array}
$$

- Both LPs are feasible. Why?
- By strong duality, dual LP has the same optimal value as the primal. What is this value? Where does it occur?

Note: The coefficients the yield the dual (or primal) optimal solution gives a certificate of optimality for the primal (or dual) solution

$$(a^*, b^*, c^*) = (1,0,1)$$

$$(x^*, y^*) = (4,4)$$

Primal

$$1 \cdot y \le 1 \cdot 4$$
$$0 \cdot x \le 0 \cdot 5$$
$$1 \cdot (x + y) \le 1 \cdot 8$$

$$\implies x + 2y \le 12$$

$\implies$ 12 is optimal

Dual

$$4 \cdot (b + c) \ge 4 \cdot 1$$
$$4 \cdot (a + c) \ge 4 \cdot 2$$

$$\implies 4a + 4b + 8c \ge 12$$
$$\implies 4a + 5b + 8c \ge 12 \quad (b \ge 0)$$

$\implies$ 12 is optimal

Complementary Slackness

Definition

Theorem: Let $\vec{\tilde{x}}$ and $\vec{\tilde{y}}$ be feasible solutions to the primal and dual, respectively. Then $\vec{\tilde{x}}$ and $\vec{\tilde{y}}$ are optimal solution if and only if **complementary slackness** holds:

- For each primal variable $\tilde{x}_i$, either $\tilde{x}_i = 0$, the corresponding dual constraint is tight, or both.
- For each dual variable $\tilde{y}_j$, either $\tilde{y}_j = 0$, the corresponding primal constraint is tight, or both.

Complementary Slackness

Illustrating Example

Primal	
max	$x + 2y$
subject to	$y \leq 4$
	$x \leq 5$
	$x + y \leq 8$

Dual	
min	$4a + 5b + 8c$
subject to	$b + c \geq 1$
	$a + c \geq 2$
	$a, b, c \geq 0$

Let us verify the optimality of solutions $(x*, y*) = (4,4)$ and $(a*, b*, c*) = (1,0,1)$ for the worked example:

- primal variable $x* = 4 \neq 0$, but dual constraint $b + c \geq 1$ is tight since $b* + c* = 1$
- primal variable $y* = 4 \neq 0$, but dual constraint $a + c \geq 2$ is tight since $a* + c* = 2$
- dual variable $a* = 1 \neq 0$, but primal constraint $y \leq 4$ is tight as $y* = 4$
- primal constraint $x \leq 5$ is not tight as $x* = 4$, but dual variable $b* = 0$
- dual variable $c* = 1 \neq 0$, but primal constraint $x + y \leq 8$ is tight as $x* + y* = 8$

Complementary Slackness

Proof Sketch

- By **strong duality,** $\vec{c}^T\vec{\tilde{x}} = \vec{b}^T\vec{\tilde{y}}$ if and only if $\vec{\tilde{x}}$ and $\vec{\tilde{y}}$ are optimal
- Rearranging, we have $\vec{\tilde{x}}^T(A^T\vec{\tilde{y}} - \vec{c}) = 0$ and $(A\vec{\tilde{x}} - b)^T\vec{\tilde{y}} = 0$
- Let's intepret:
  - If any primal variable $\tilde{x}_i \neq 0$, then $(A^i)^T\vec{\tilde{y}} - c_i = 0$ (i.e., $(A^i)^T\vec{\tilde{y}} = c_i$ so *no slack*), where A^i is column i of A, so the dual constraint for $\tilde{x}_i$ is tight
  - If the slack in the dual constraint $(A_i^T\vec{\tilde{y}} - c_i) \neq 0$, then $\tilde{x}_i = 0$

Repeat for dual variables and primal constraints!