Folding One Polyhedral Metric Graph into Another

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Abstract. We analyze the problem of folding one polyhedron, viewed as a metric graph of its edges, into the shape of another, similar to 1D origami. We find such foldings between all pairs of Platonic solids and prove corresponding lower bounds, establishing the optimal scale factor when restricted to integers.

1 Introduction

Viewing a polyhedron as a metric graph (graph with specified edge lengths) [1], when can we ***fold*** it into another polyhedron, in the sense of 1D origami where lengths must be preserved and we view multiple overlapping layers as one? More formally:

Problem 1. Given two metric spaces A and B, find an ***isometric covering*** of B from A, that is, a surjective map $m : A \to B$ such that, for every path p in A, the arc length of p in A equals the arc length of $m(p)$ in B. Figure 1 shows an example.

To ensure this is always possible, we can scale the lengths in A by a constant ***scale factor*** α (in Figure 1, $\alpha = 7/4$ for unit edge lengths), and aim to minimize α. First we prove this problem NP-complete and hard to approximate within a constant factor (Section 2). Then we analyze optimal mappings between pairs of Platonic solids, finding mappings to establish upper bounds (Section 3) and proving lower bounds (Section 4). Table 1 summarizes these results, which are tight if restricted to integral α.

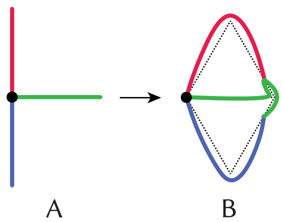

Fig. 1. Example folding.

Table 1. Our integer-tight lower and upper bounds, given as intervals, on the minimum scale factor for folding one Platonic solid (row) into another (column), both with unit edge lengths.

↱	Tetrahedron	Cube	Octahedron	Dodecahedron	Icosahedron
Tetrahedron	[1, 1]	[2+1/3, 29/10]	(2, 2+1/2]	[6+1/3, 6+2/3]	[5+2/3, 5+7/8]
Cube	(1/2, 5/6]	[1, 1]	(1, 1+1/2]	[2+1/2, 3]	[2+1/2, 3]
Octahedron	(1/2, 1]	[1+1/12, 1+1/2]	[1, 1]	[3+1/12, 4]	[2+1/2, 3]
Dodecahedron	(1/5, 3/5]	(2/5, 4/5]	(2/5, 3/4]	[1, 1]	(1, 1+1/3]
Icosahedron	(1/5, 1]	(1, 1+1/3]	(2/5, 1]	[1+2/15, 2]	[1, 1]

2 Hardness

Theorem 1 (Inapproximability). *Given two metric planar graphs G_1 and G_2, deciding whether G_1 can be folded onto G_2 via an isometric covering is NP-complete, and the optimal scale factor OPT of mapping G_1 onto G_2 is NP-hard to approximate within a factor of* < 1.5.

3 Platonic Upper Bounds (Foldings)

To achieve the upper bounds reported in Table 1, we developed initial solutions by observing inscriptions of one polyhedron in another (Figure 2a), and then further optimized these solutions through manual rerouting and automated search. For the latter, we combined two brute-force paradigms—logic programming and integer linear programming—along with local improvement techniques.

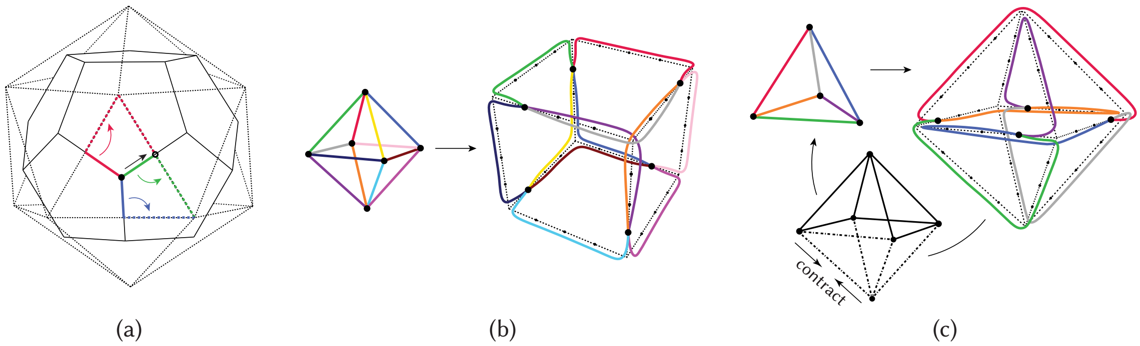

Fig. 2. (a) Mapping an icosahedron to a dodecahedron via an inscription. Routing edges via subdivisions of (b) an octahedron to a cube with $\alpha = 3/2$, and (c) a tetrahedron to an octahedron with $\alpha = 5/2$. Mappings are not drawn to scale.

4 Lower Bounds

For any pair of polyhedra, a naïve lower bound on the optimal scale factor is immediate: every target edge needs to be covered by a source edge, thus $\mathrm{OPT} \geq \frac{\text{perimeter of target graph}}{\text{perimeter of source graph}}$. However, we significantly improve this lower bound with the following observations:

- Each source vertex is mapped to either a vertex or a point along an edge of the target;
- Each source edge is routed to a path in the target;
- The scale factor is the maximum length of the routed target paths.

Let n_s be the number of vertices in the source graph, and let o_t be the number of vertices of odd degree in the target. Then:

Lemma 1. *In any solution, at least $\frac{o_t - n_s}{2}$ target edges must be fully doubly covered. If at least one source vertex is placed in the middle of a target edge, then the bound becomes strict.*

The following result forms the basis of our lower bounds:

Theorem 2 (Lower Bound). $\mathrm{OPT} \geq \frac{\text{perimeter of target graph}+\text{lengths of doubly covered target edges}}{\text{perimeter of source graph}}$.

5 Future Work

Next steps include tightening the bounds in Table 1 and developing heuristics for folding general polyhedra. We also wish to strengthen the hardness proof to apply to polyhedral graphs, which requires more connectivity than our current construction. Finally, we plan to develop solutions that permit *continuous* folding motions, which are required in the case of *rigid* polyhedral linkages.

References

1. Erik D. Demaine and Joseph O'Rourke. *Geometric Folding Algorithms: Linkages, Origami, Polyhedra.* Cambridge University Press, 2007.