Graph Threading

Erik Demaine, Yael Kirkpatrick, and Rebecca Lin

ITCS 2024

Massachusetts Institute of Technology

beadwork from Beady
[Igarashi et al. 2012]

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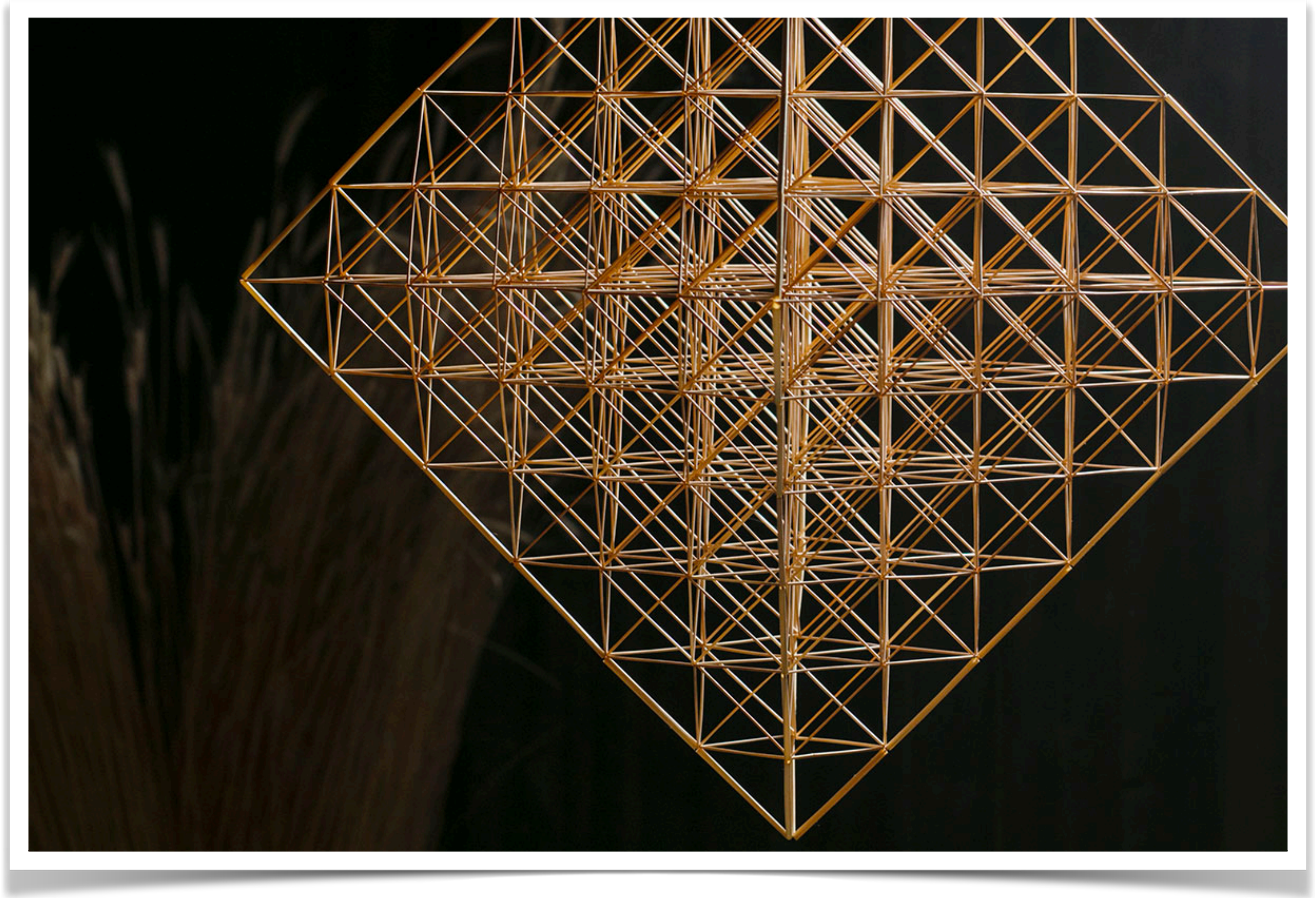

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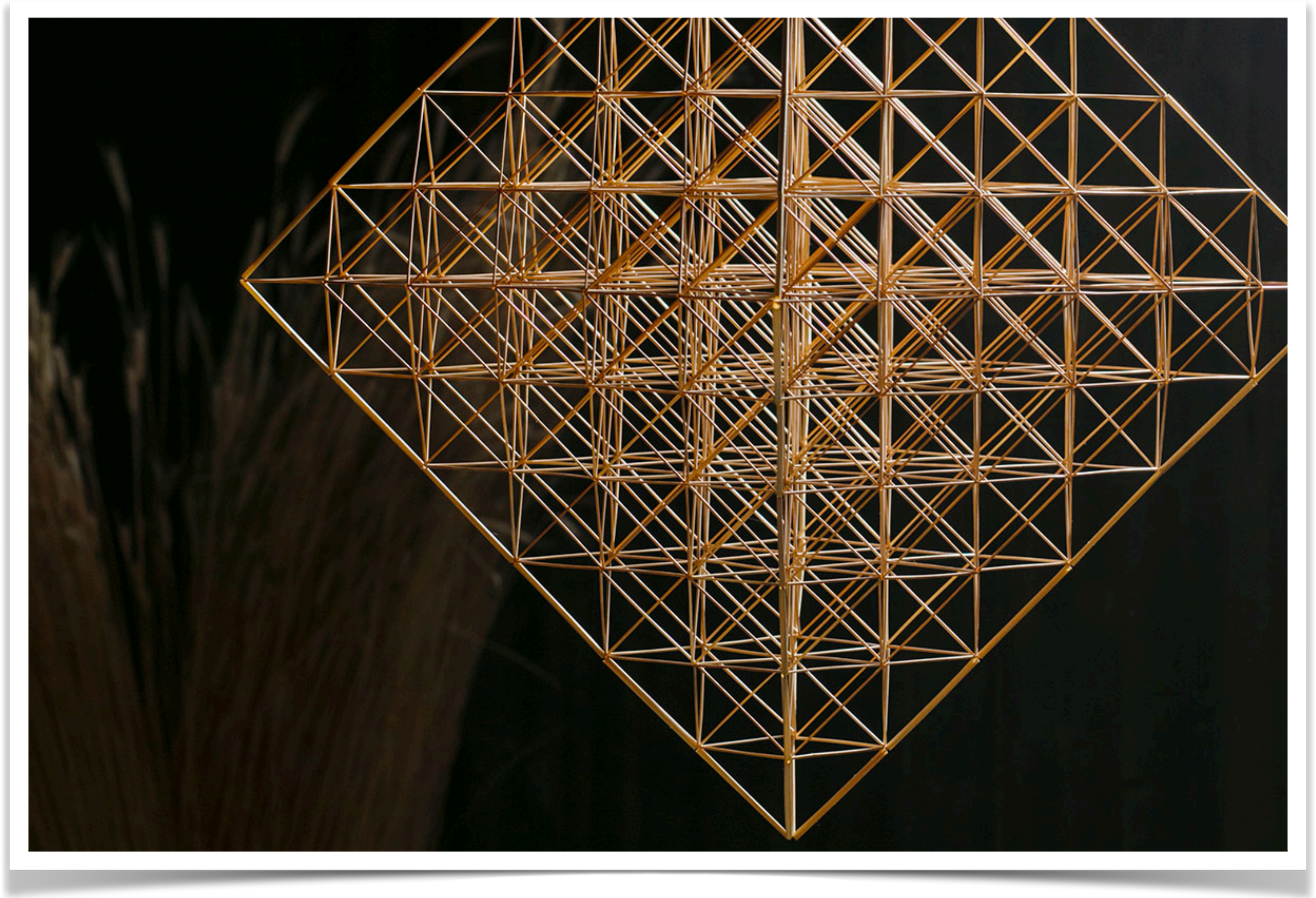

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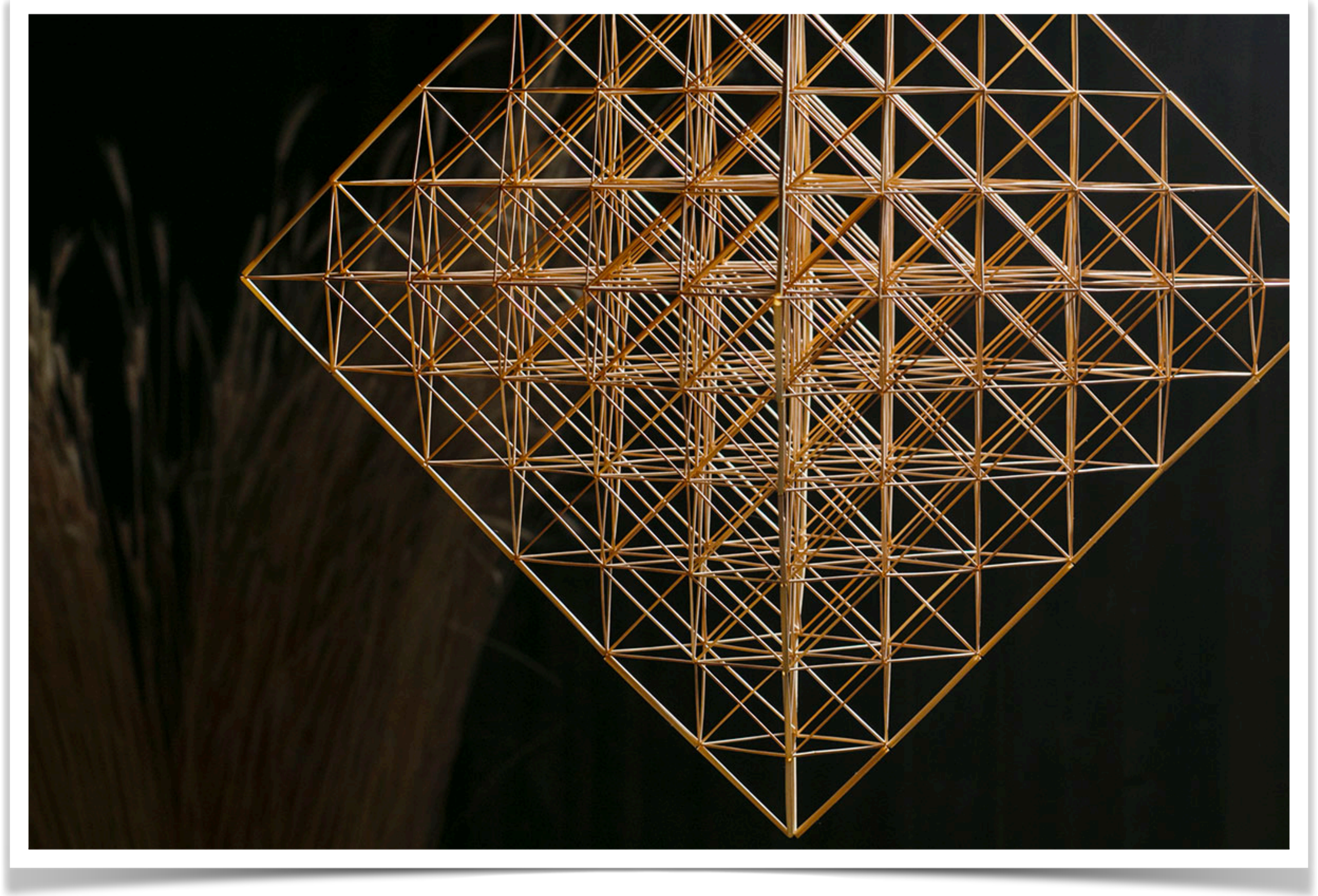

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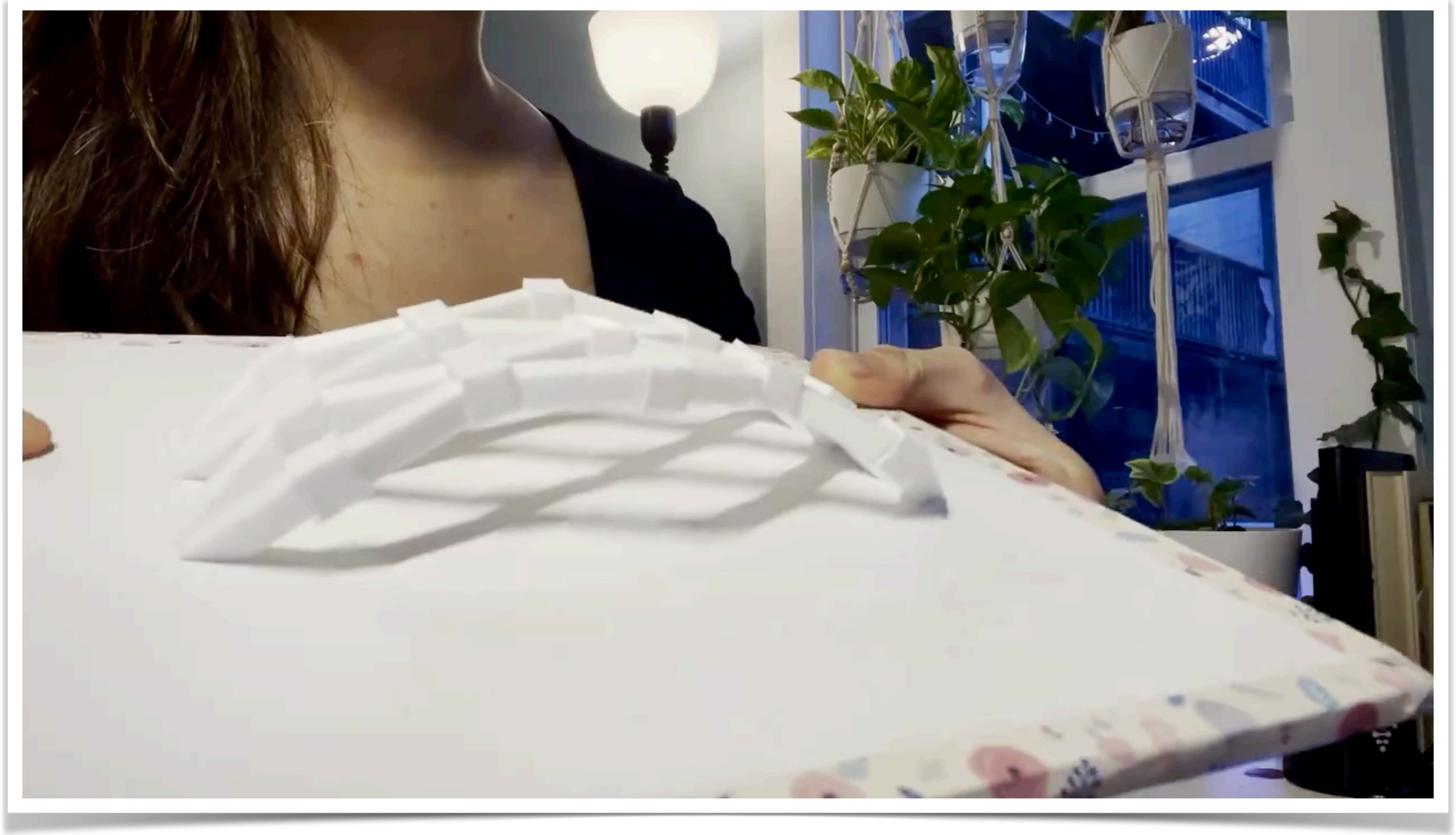
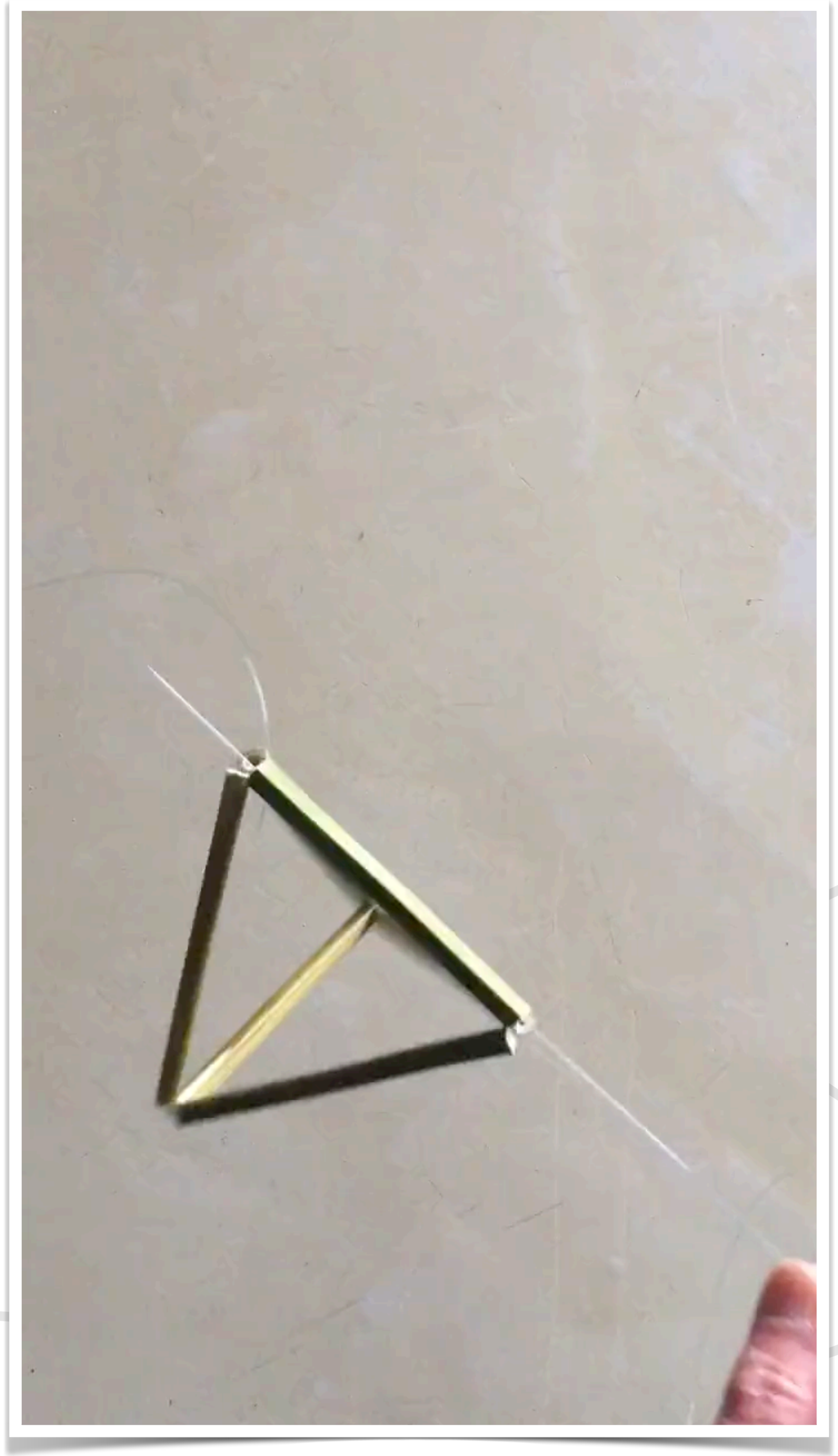


a push puppet-inspired deployable structure by Lin and Tachi

a deployable dome by Alison Martin



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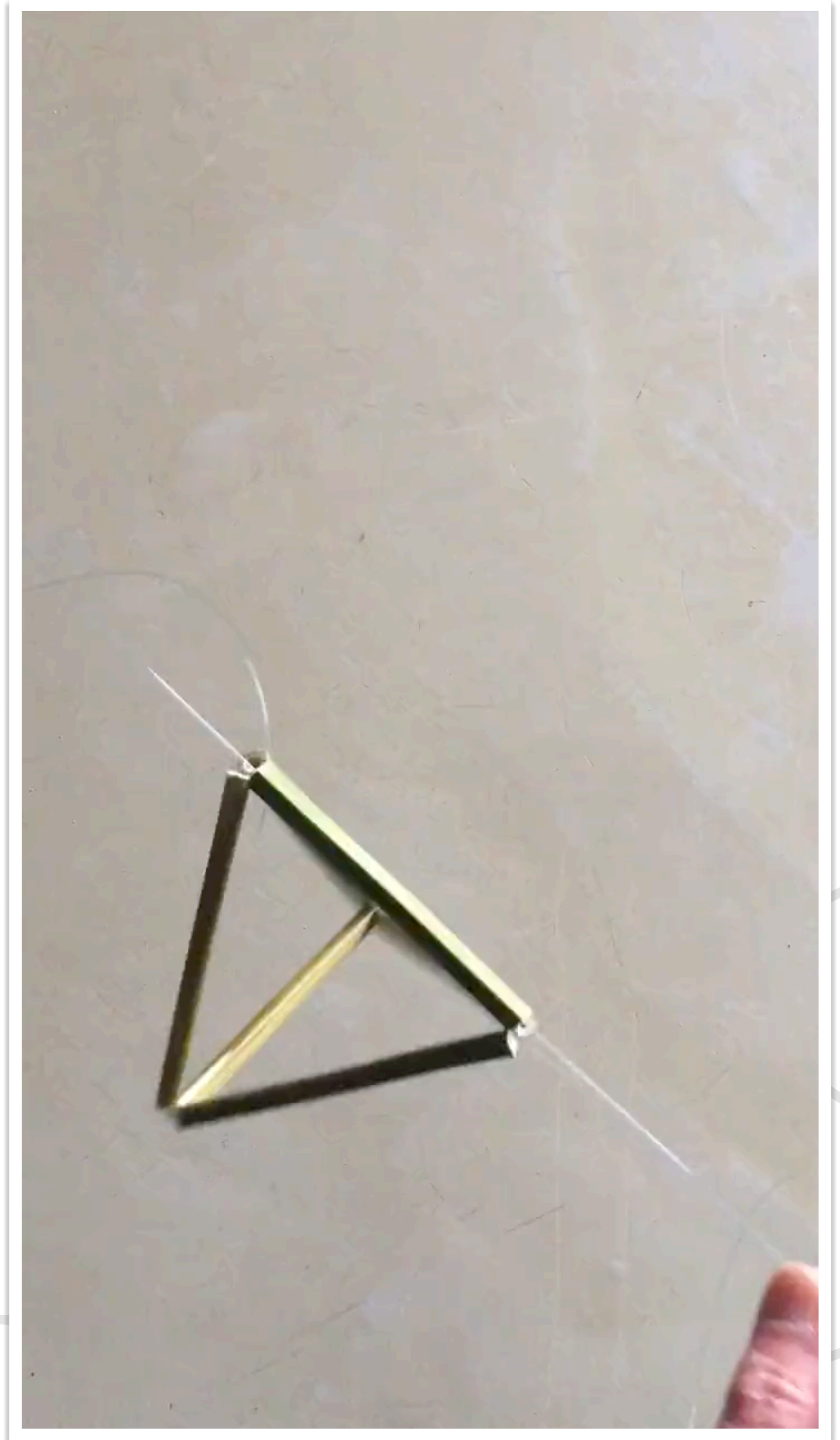

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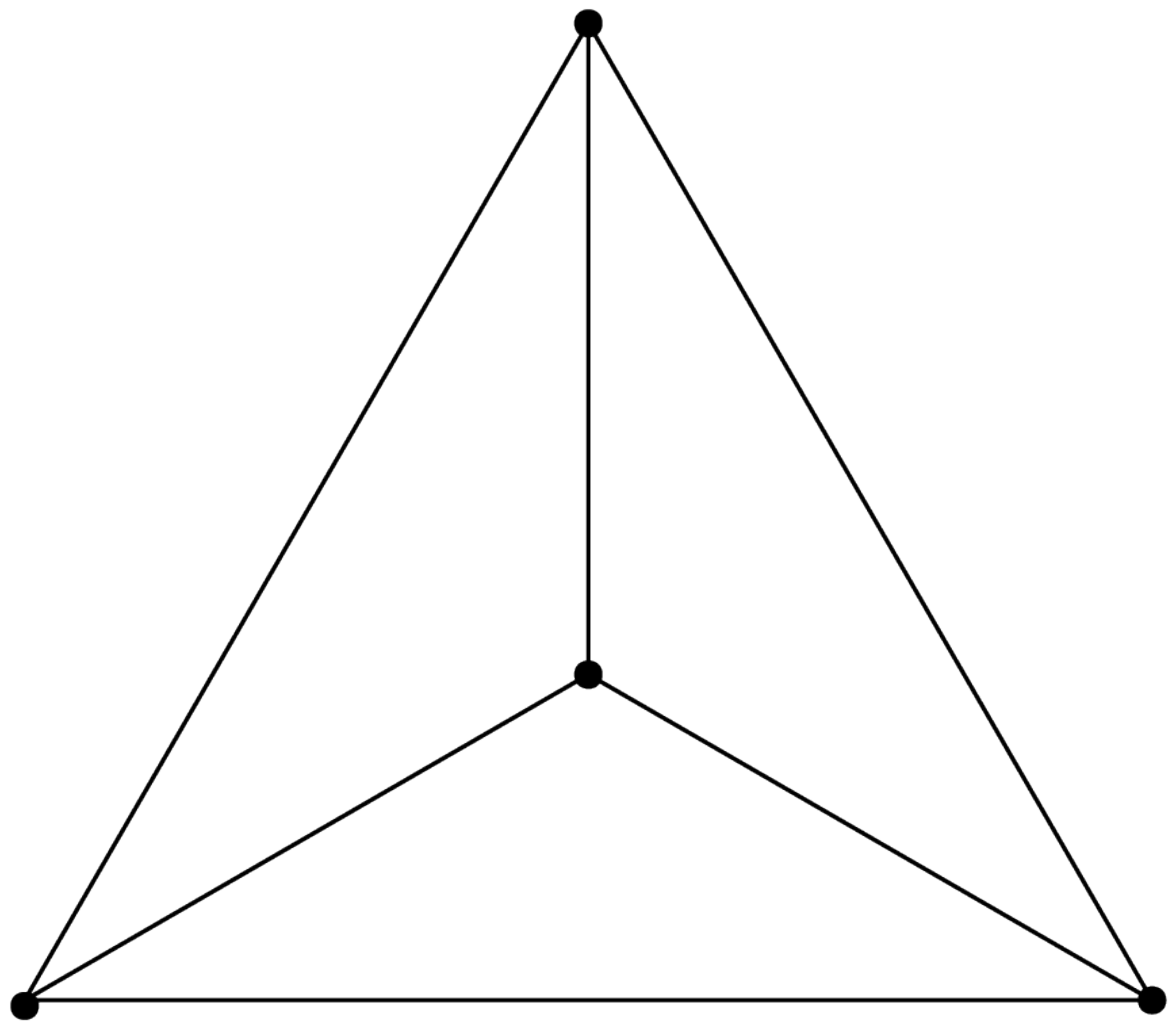

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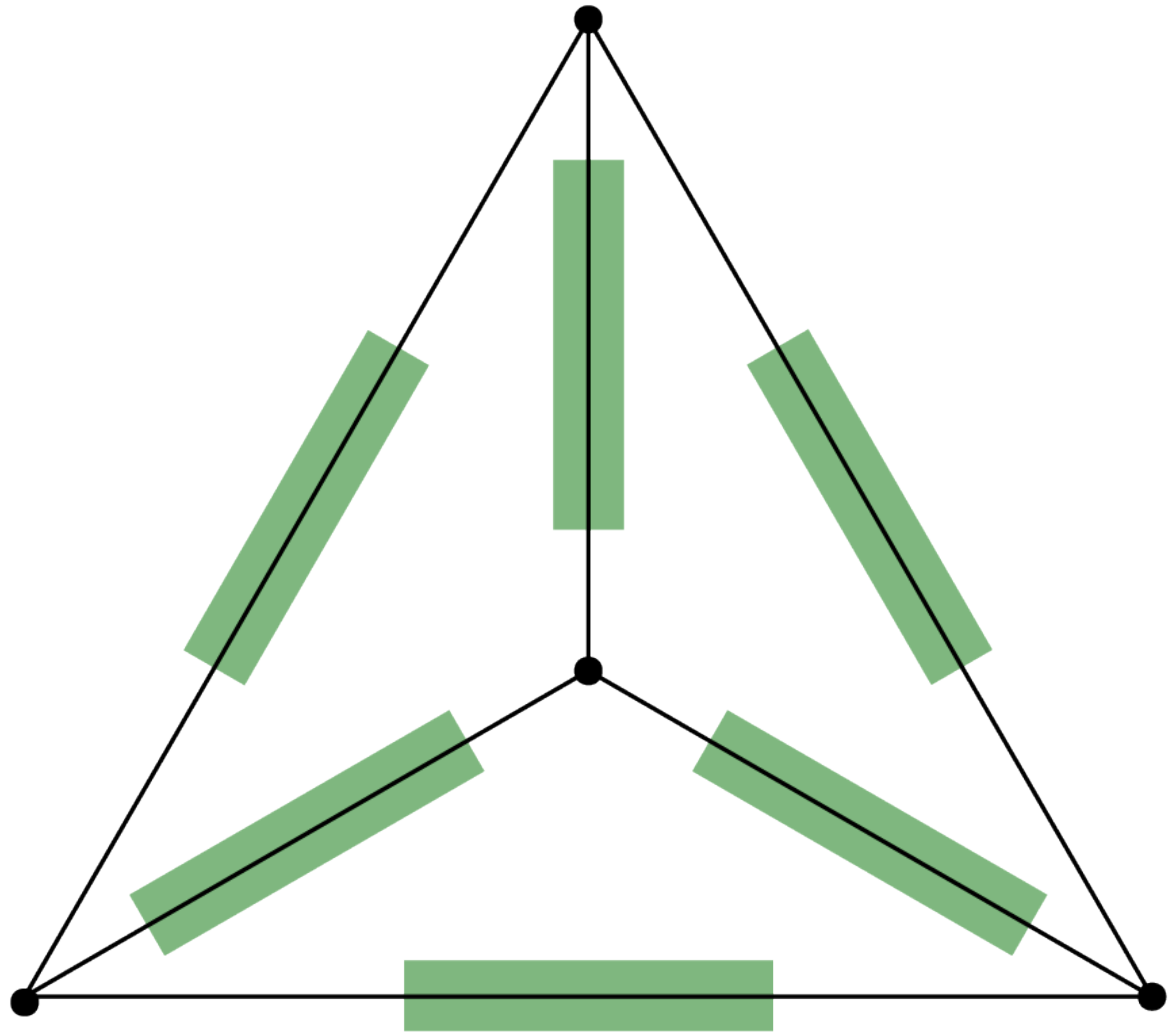

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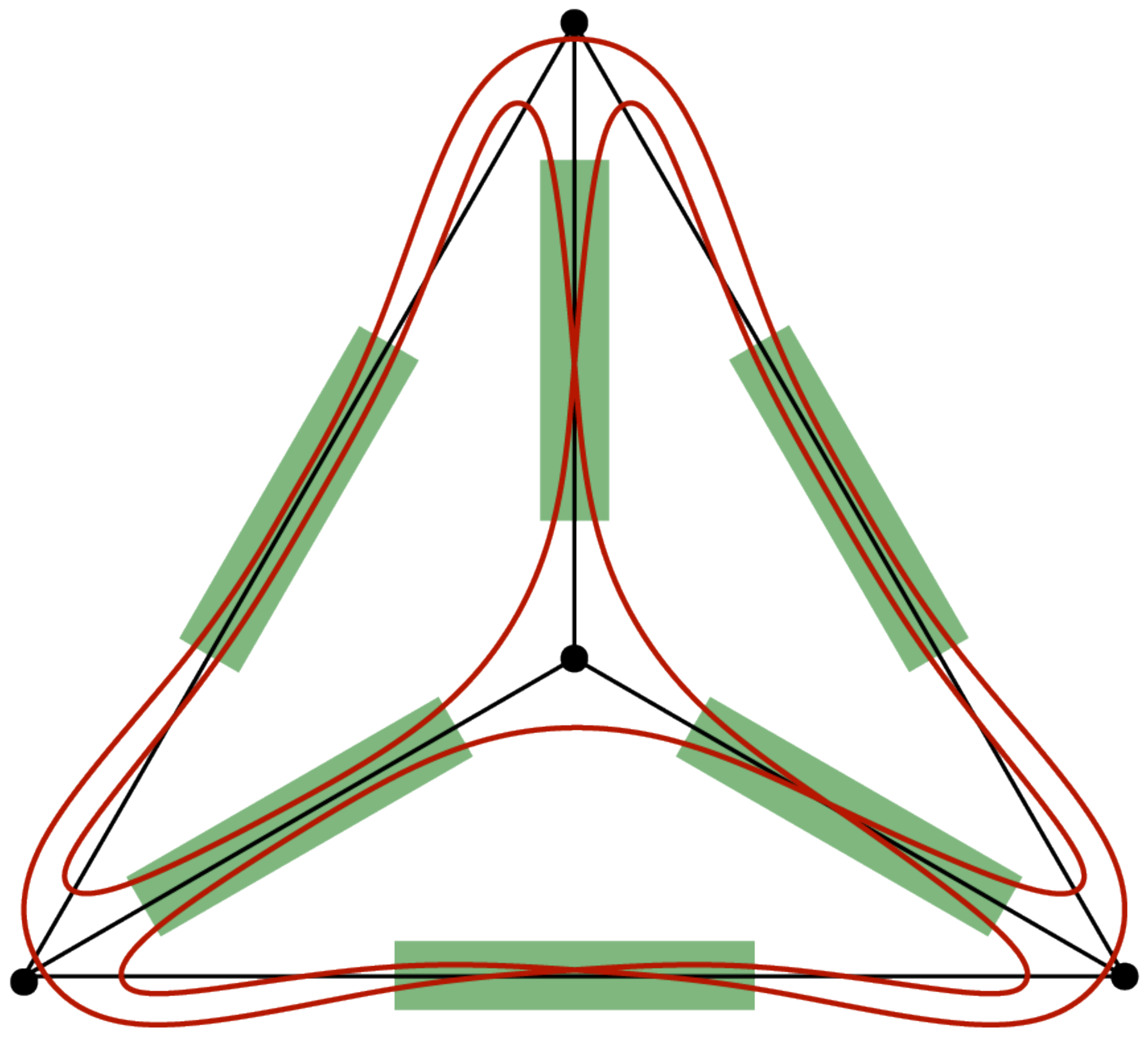

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A **threading** T of G is a closed walk through G that visits each edge at least once,
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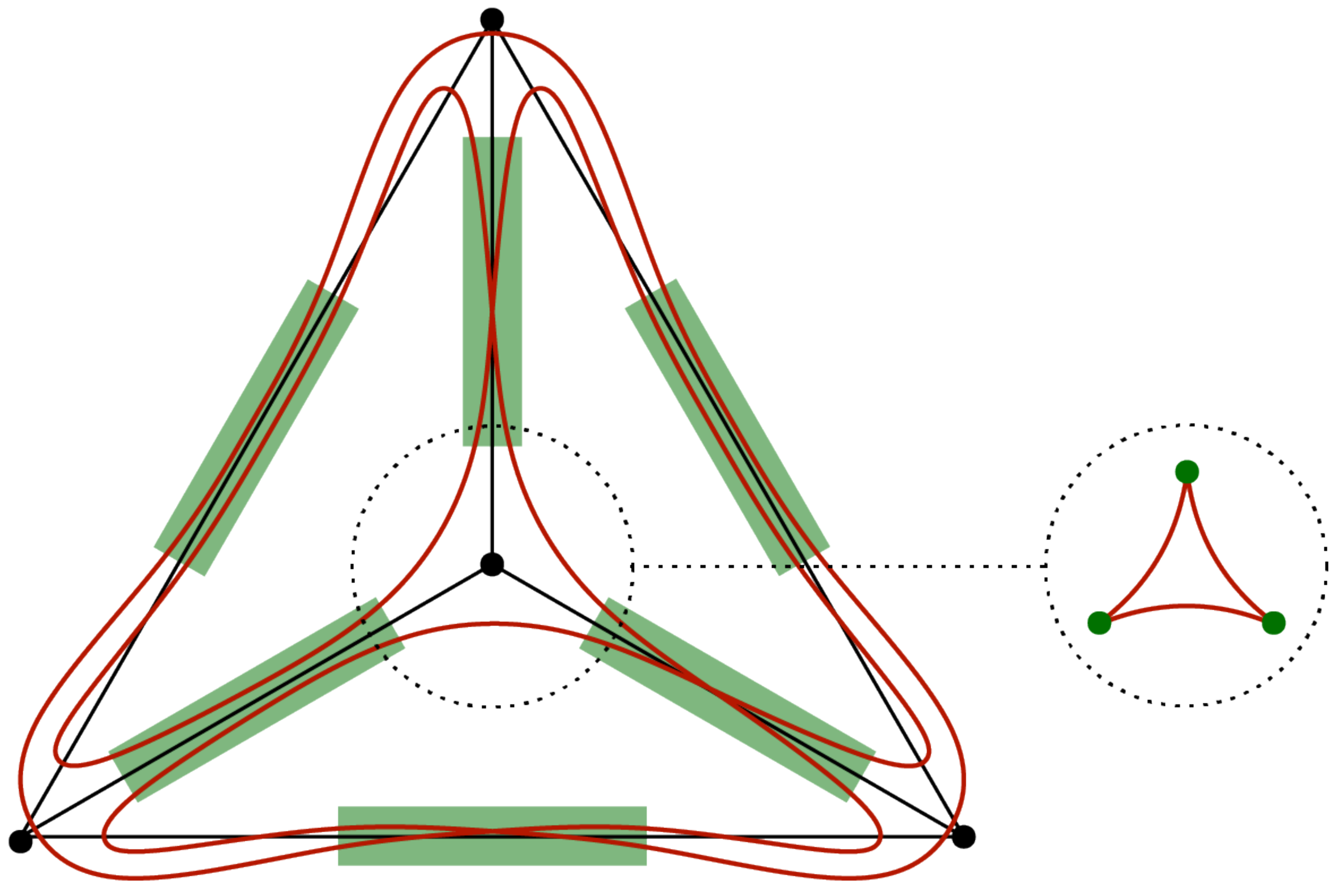

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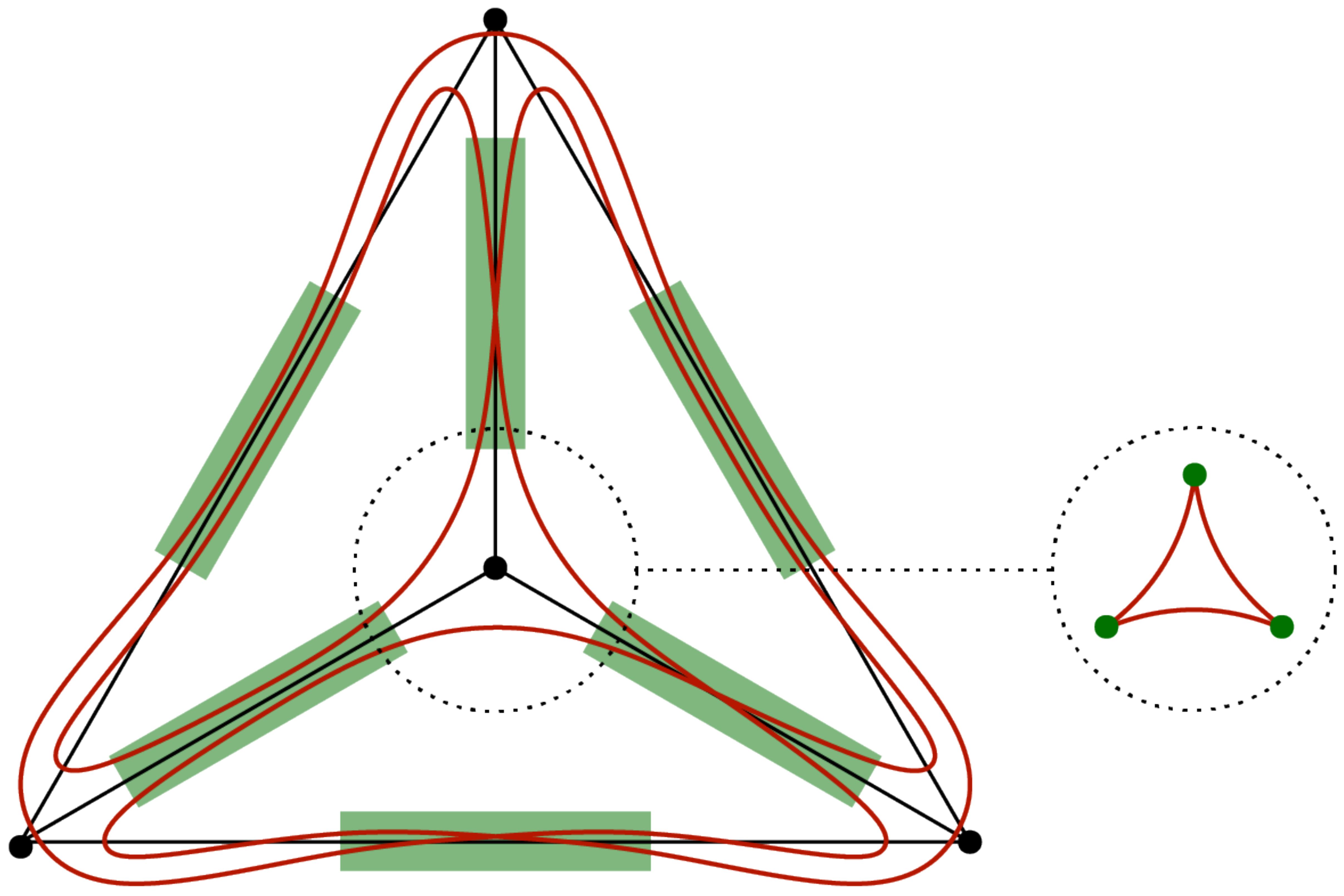

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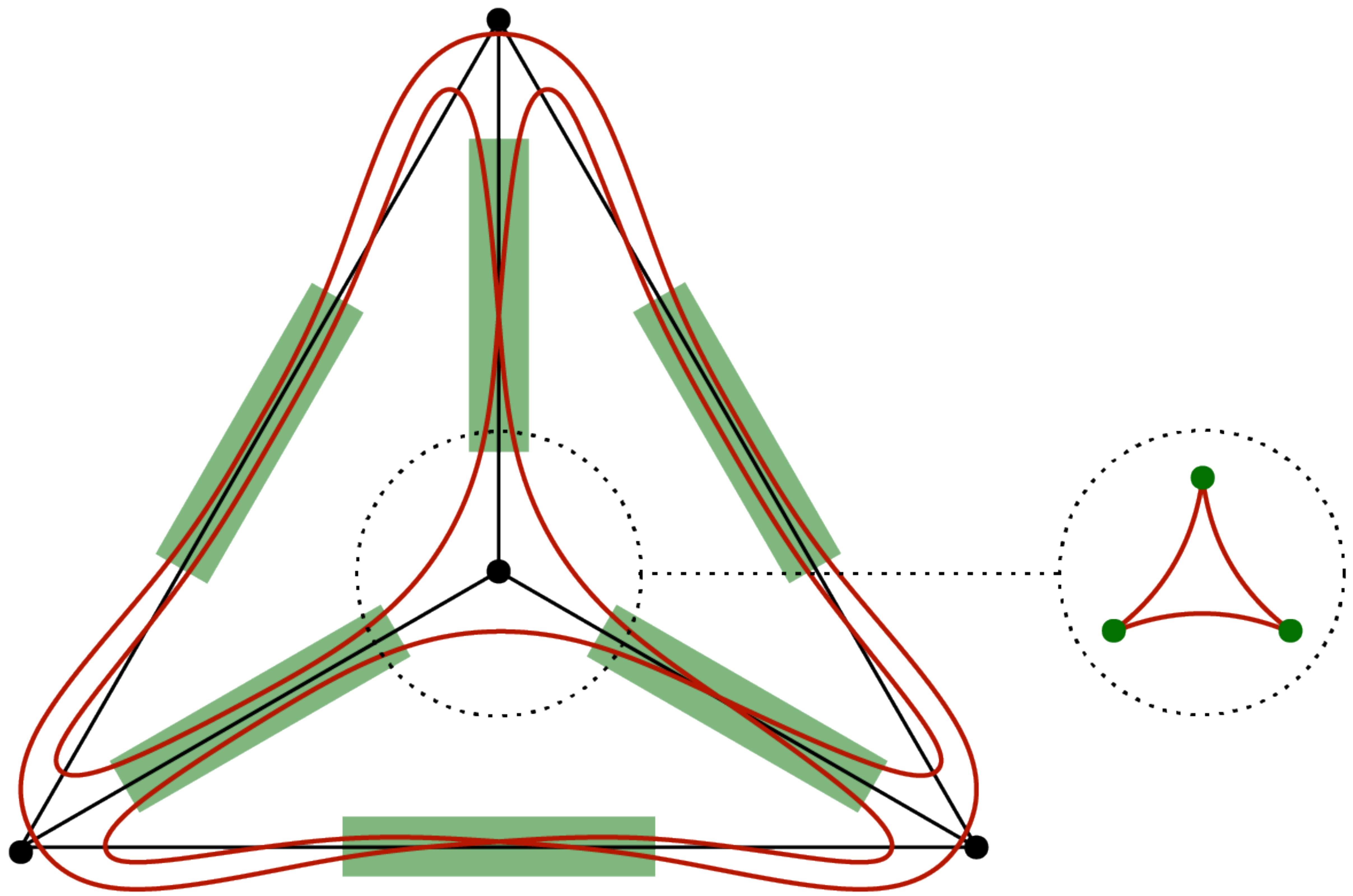

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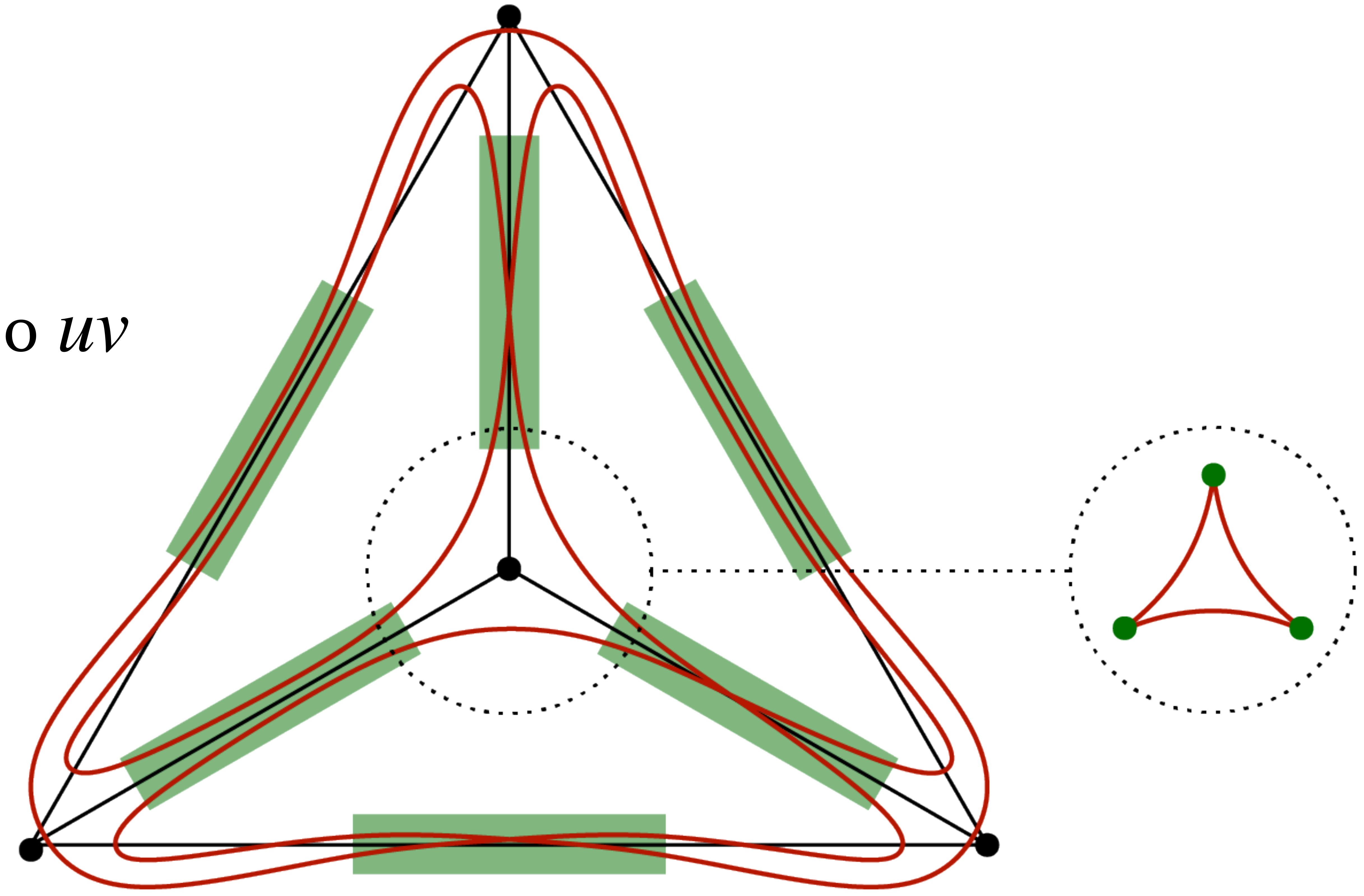

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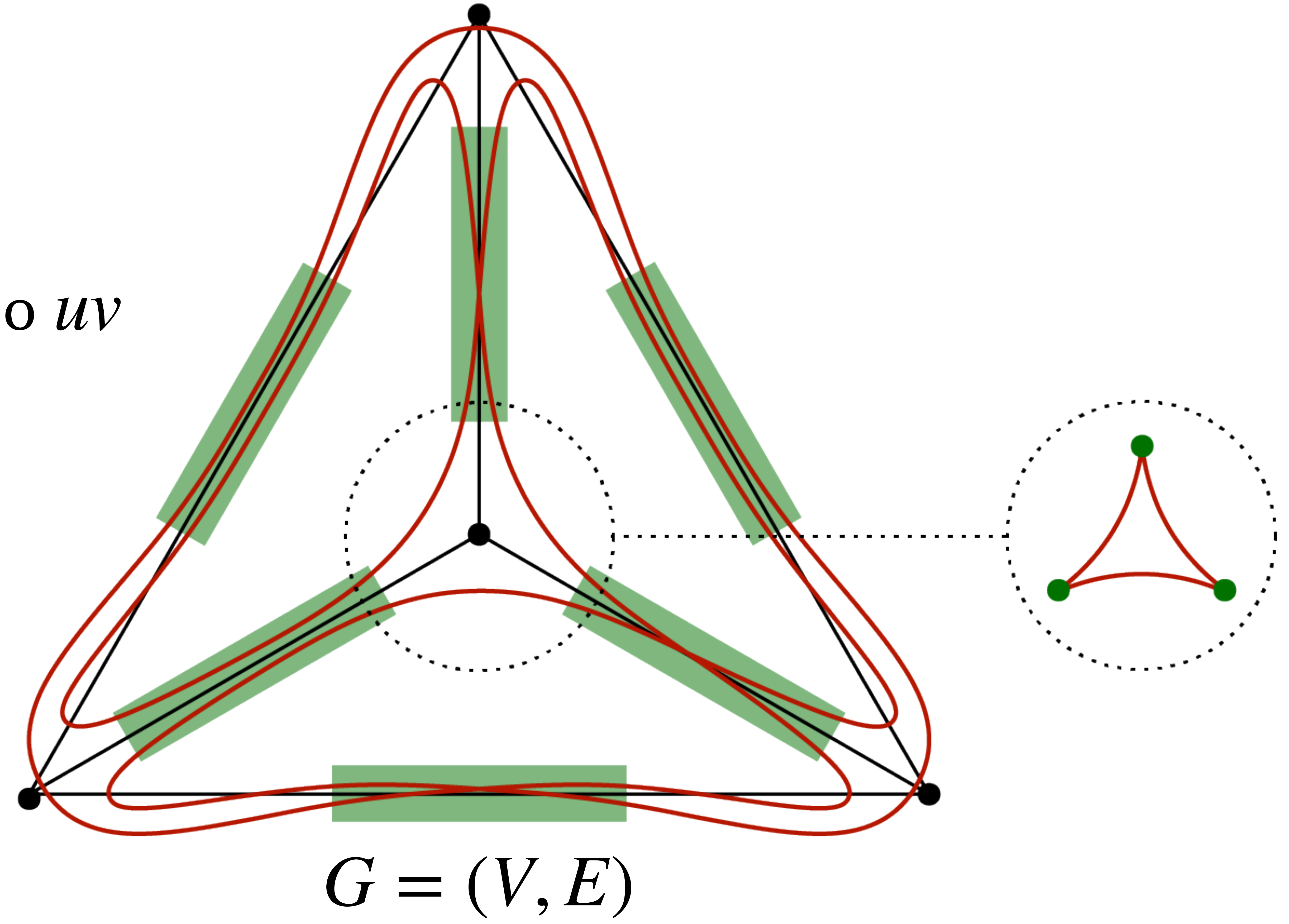
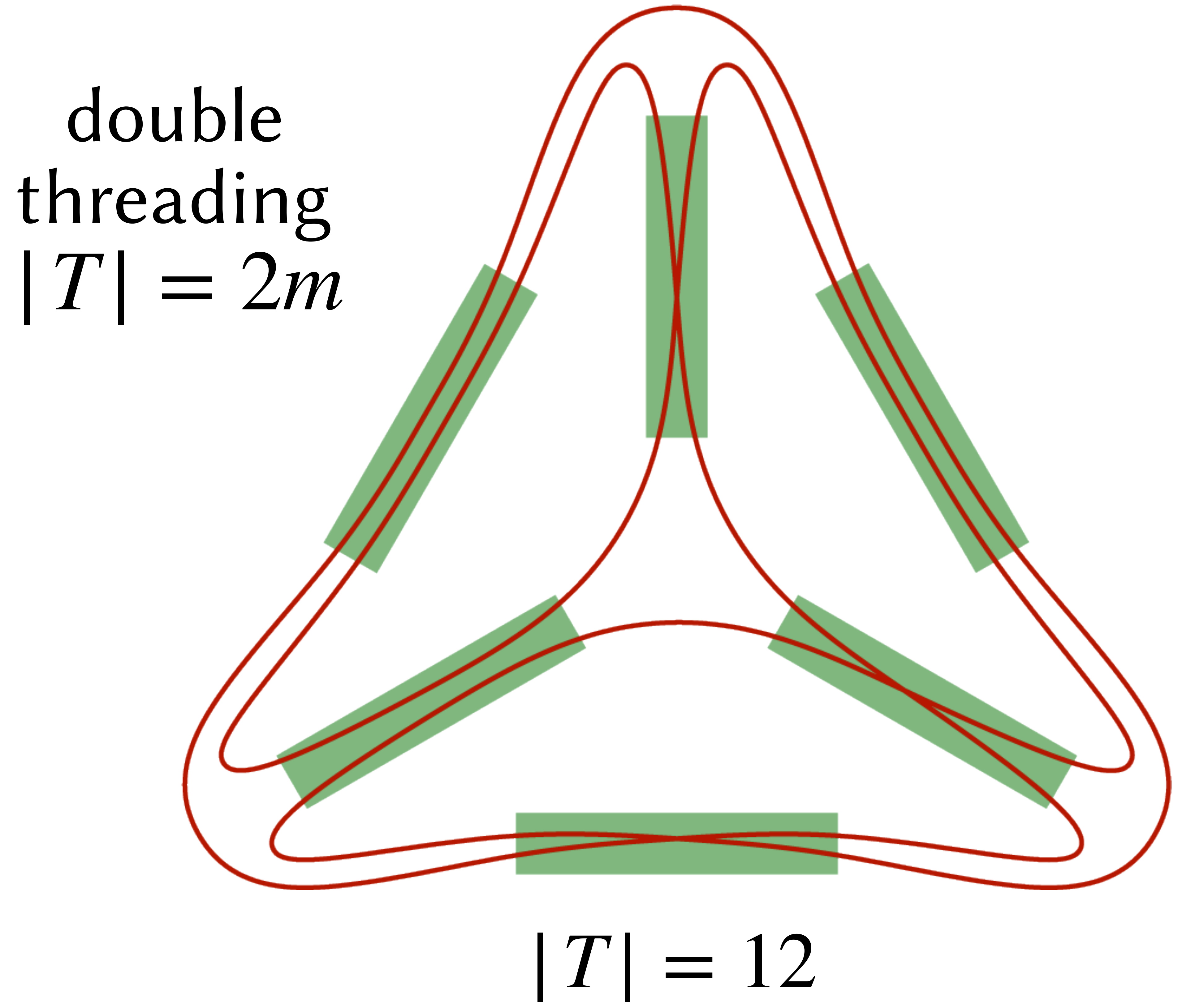
length $|T| = \sum_{uv \in E} x_{uv}$

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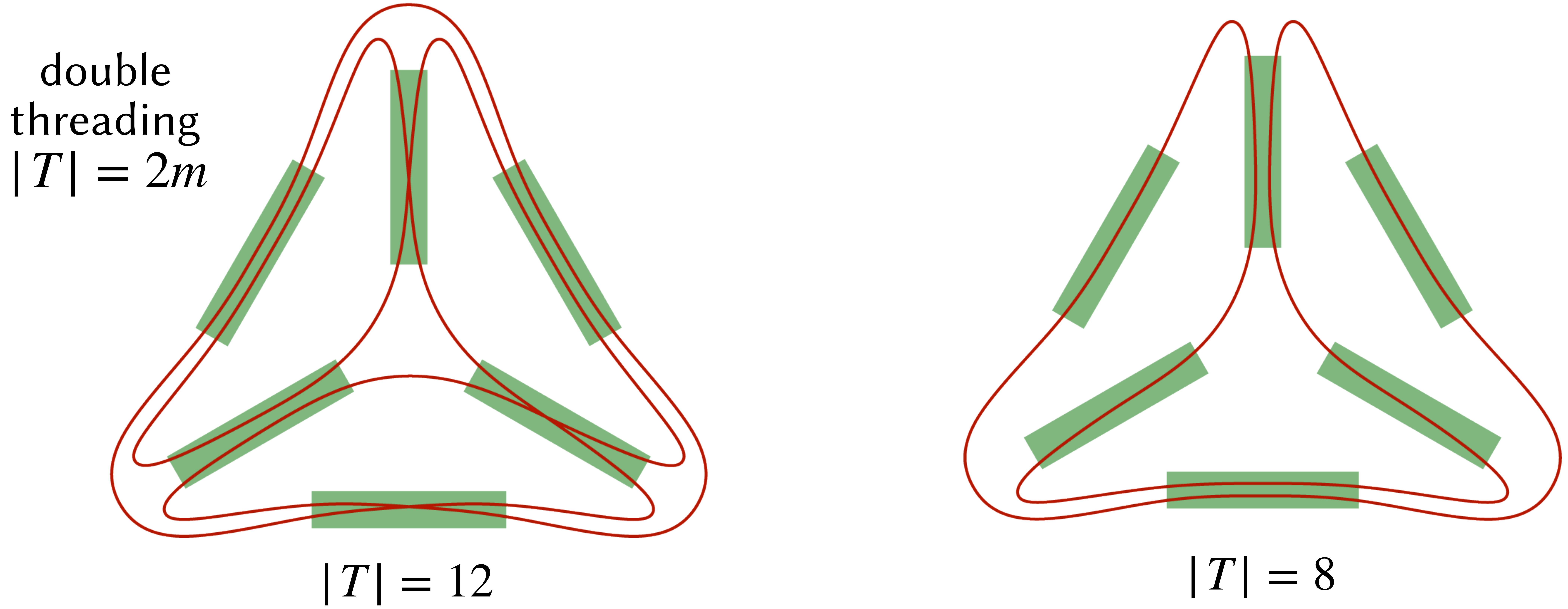
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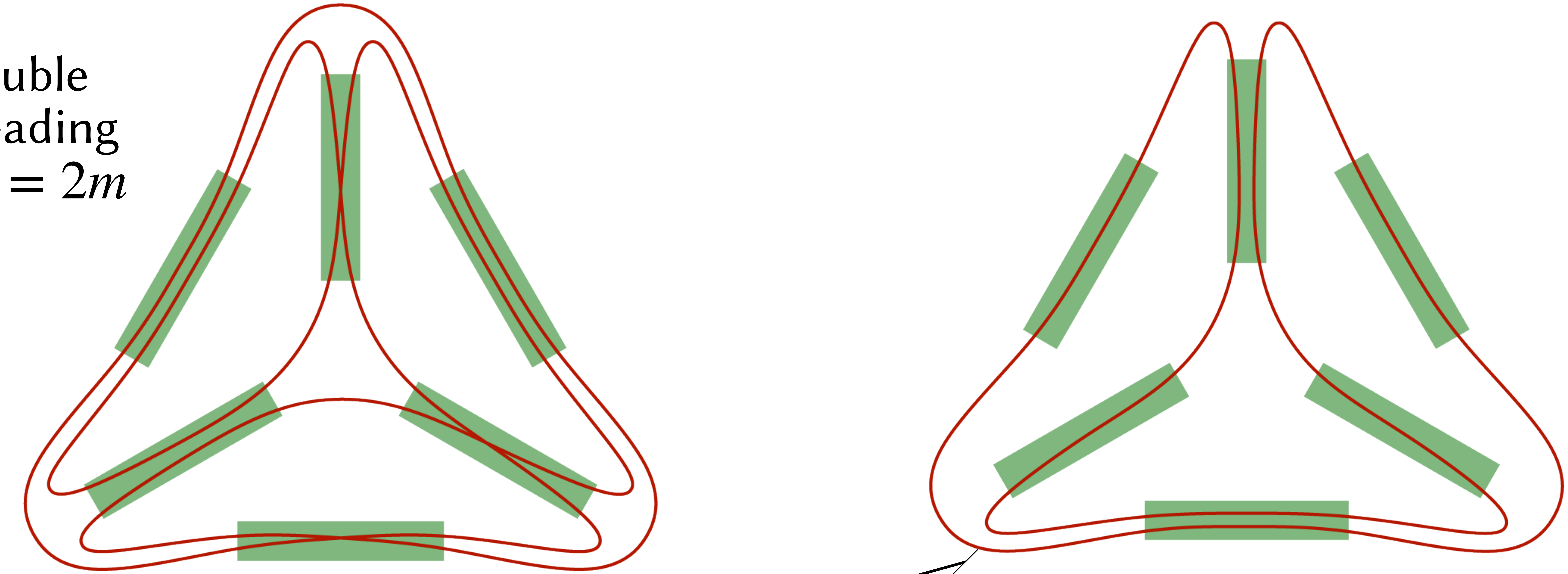

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Goal: Minimize $|T| = \sum_{uv \in E} x_{uv}$

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- Polynomial-time algorithm to compute the edge counts of an optimal threading
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Bounds:

- An optimal threading has length at least $2m - n$
- There exist graphs with optimal threadings of length $2m - O(1)$

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- Show a linear-time algorithm to **construct a global threading** from local solutions
- Give an algorithm to **compute local solutions** via reduction to perfect matching

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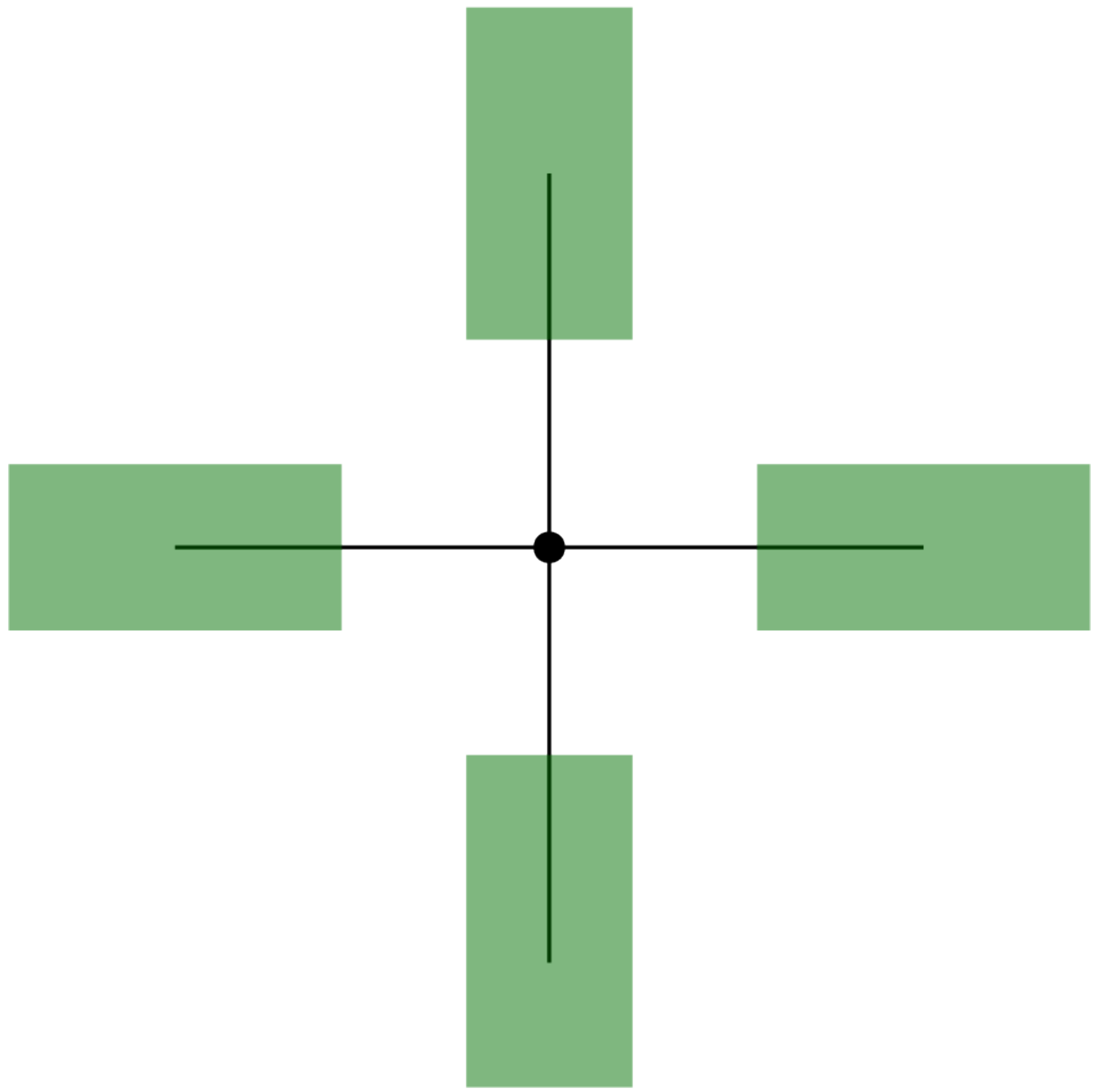

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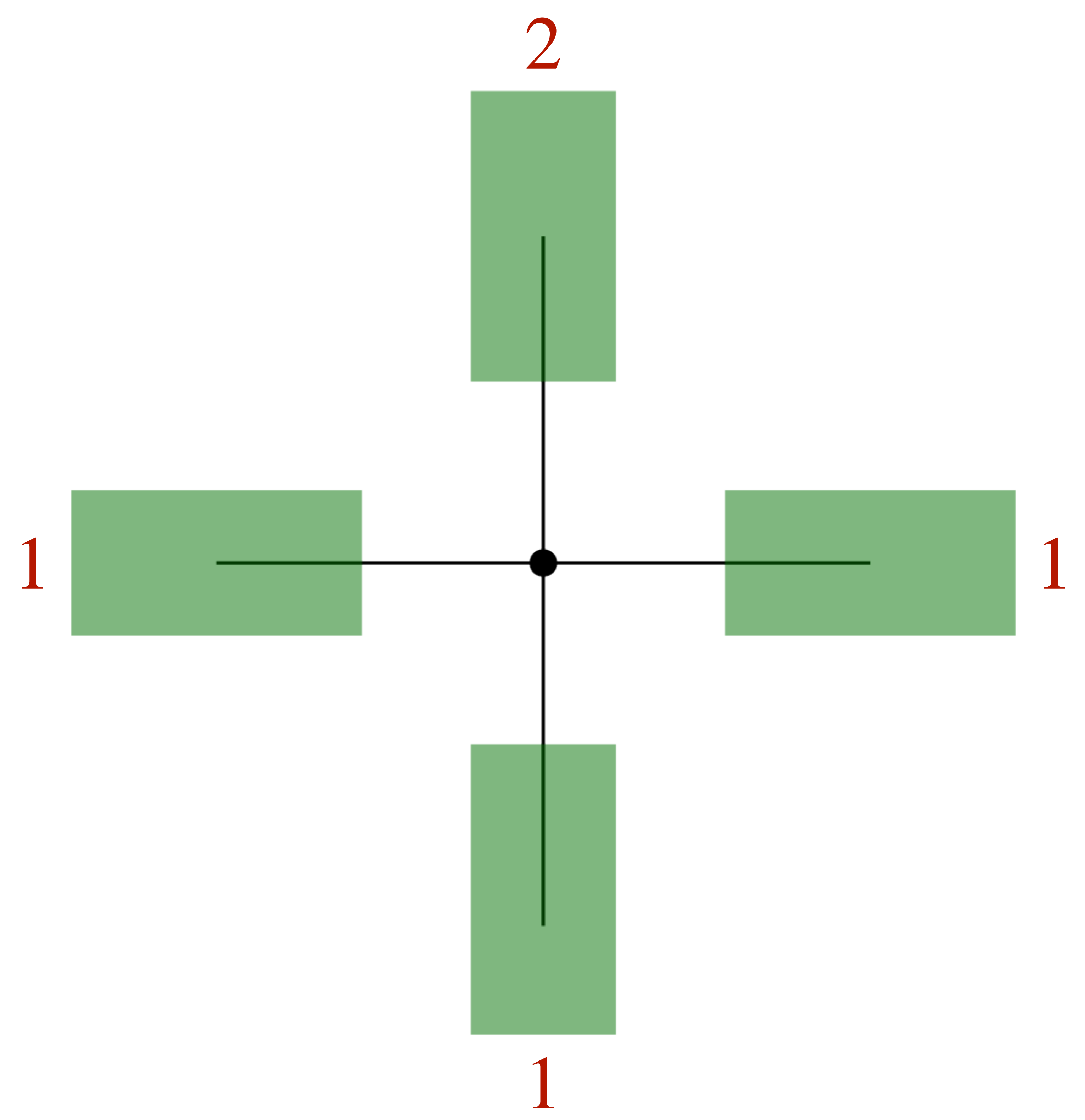

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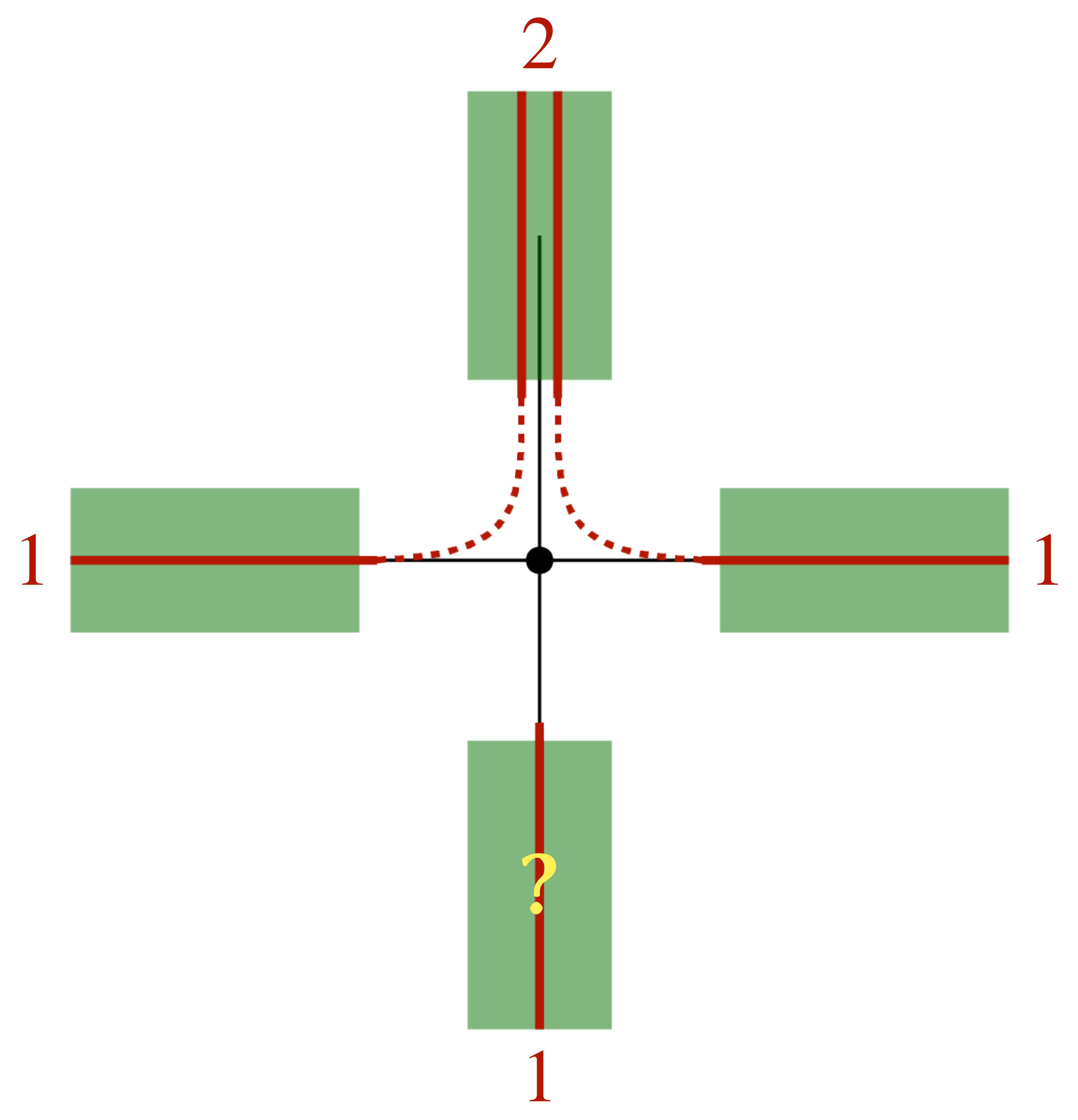

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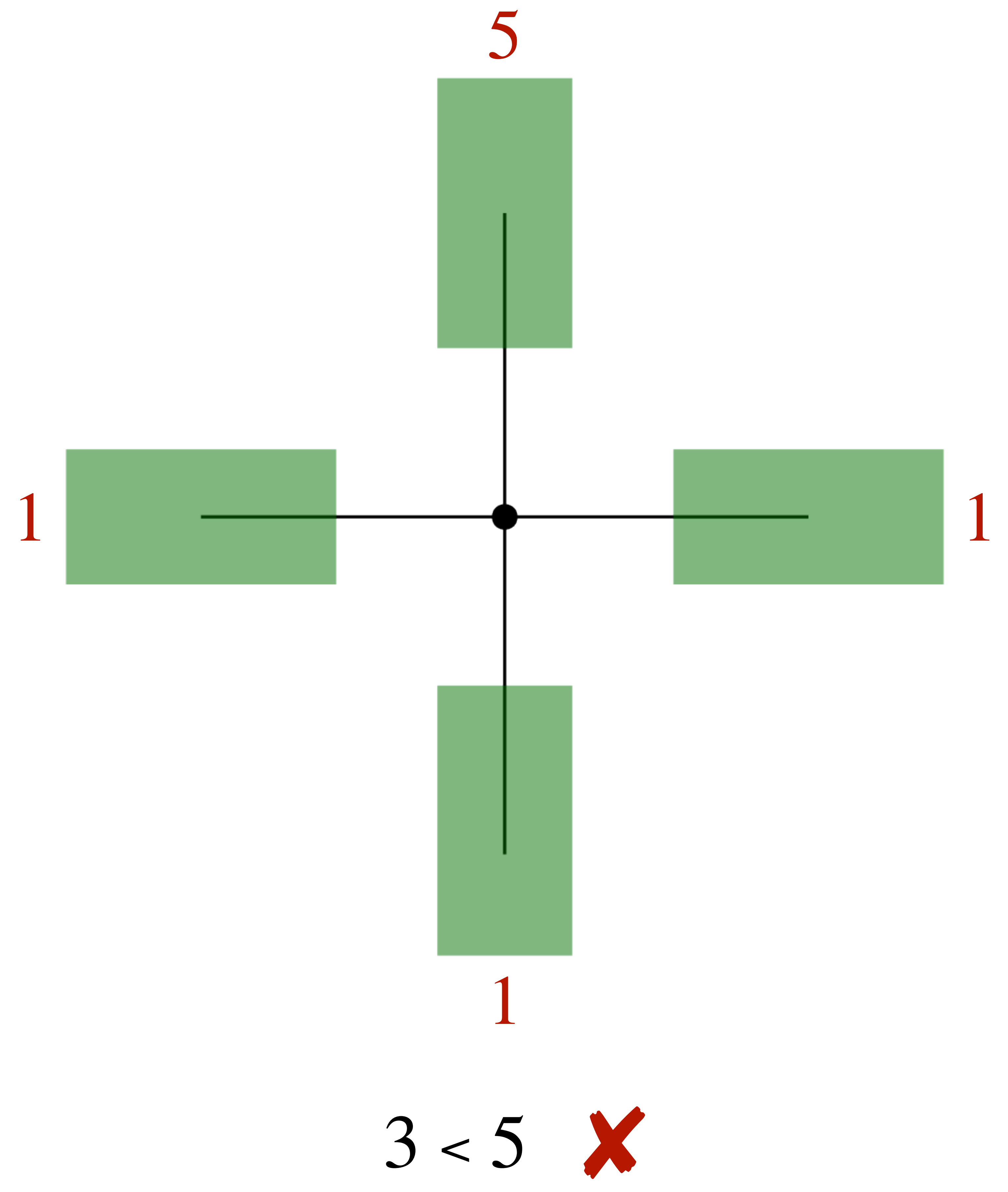

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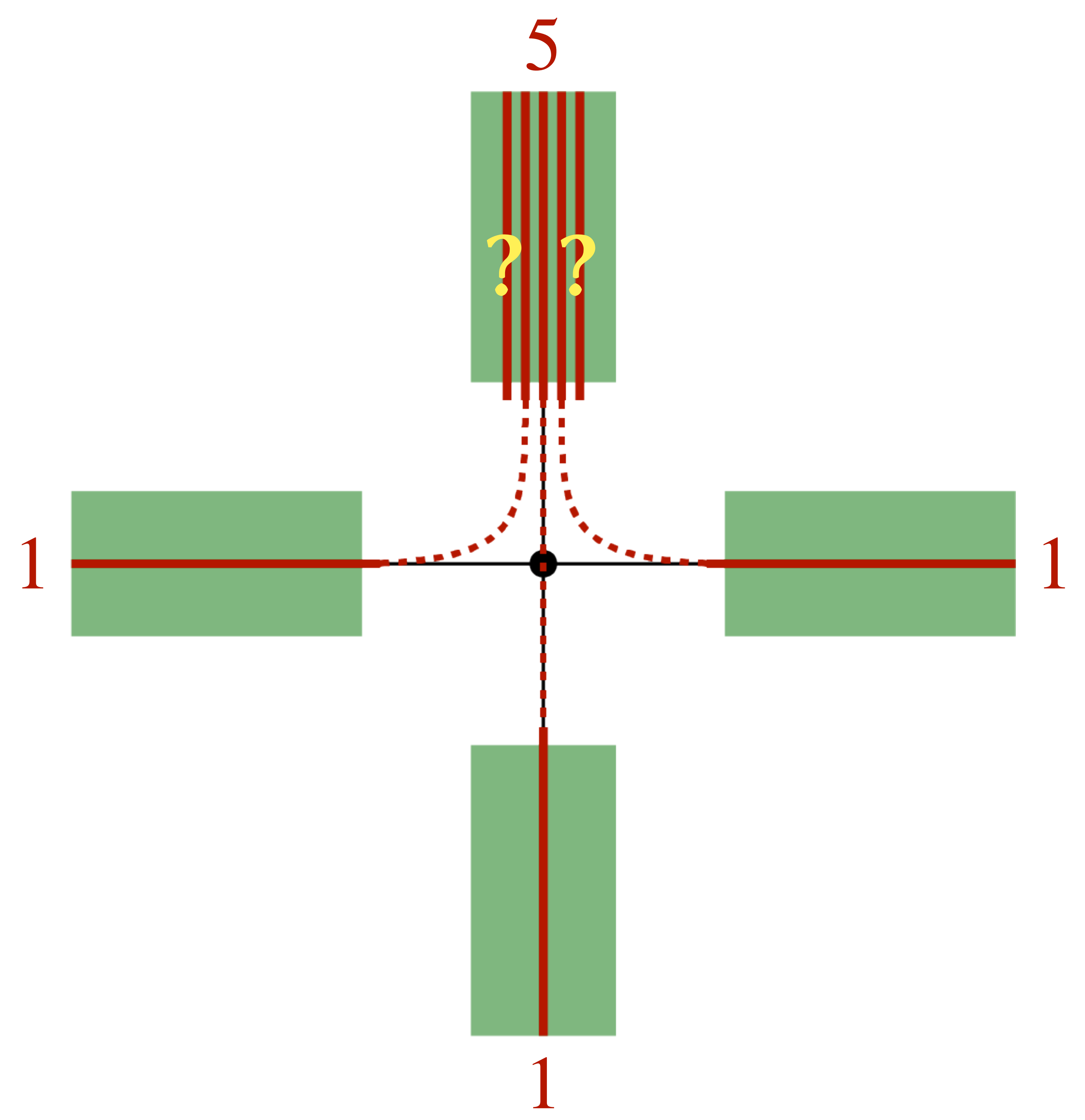

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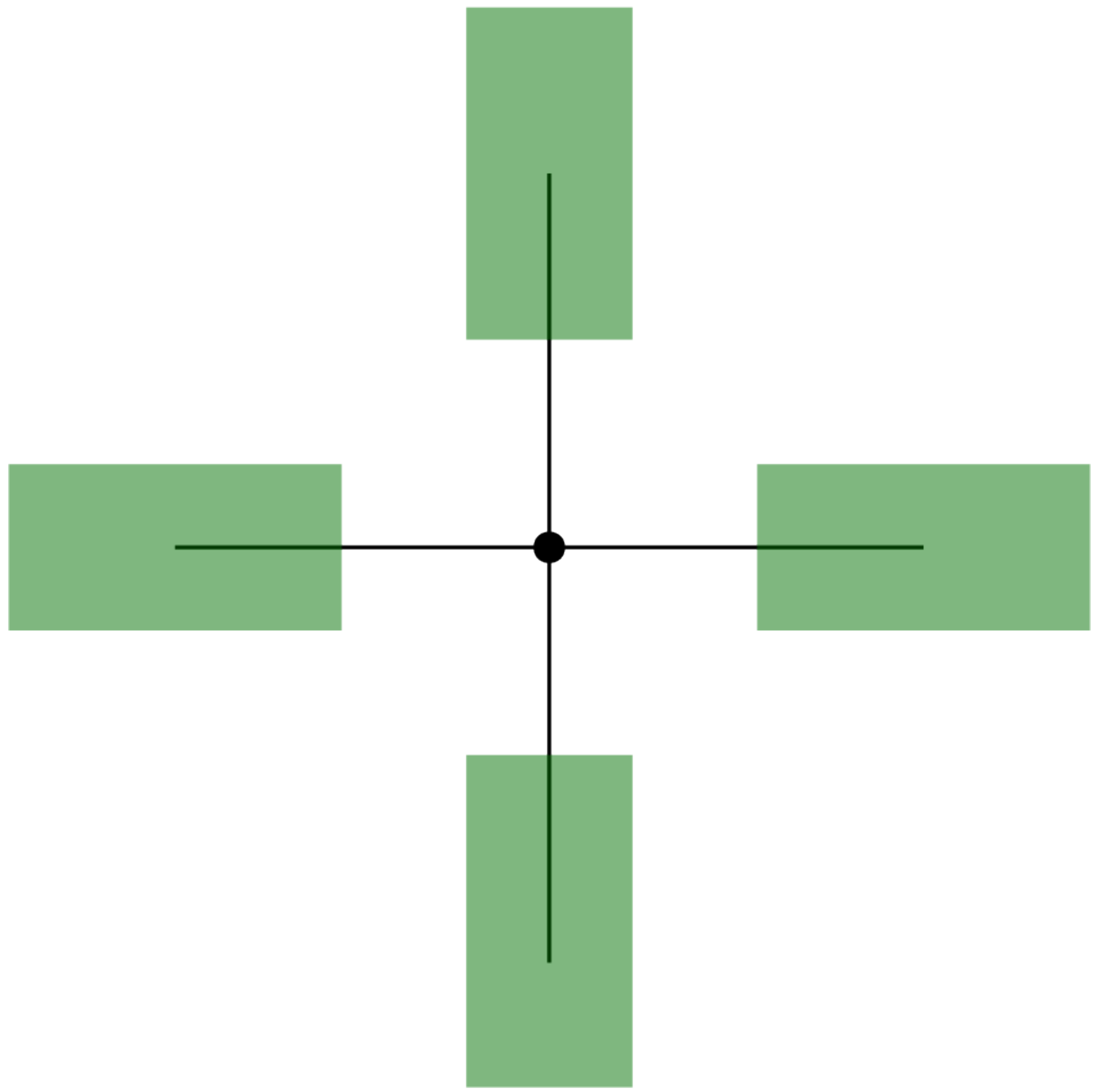

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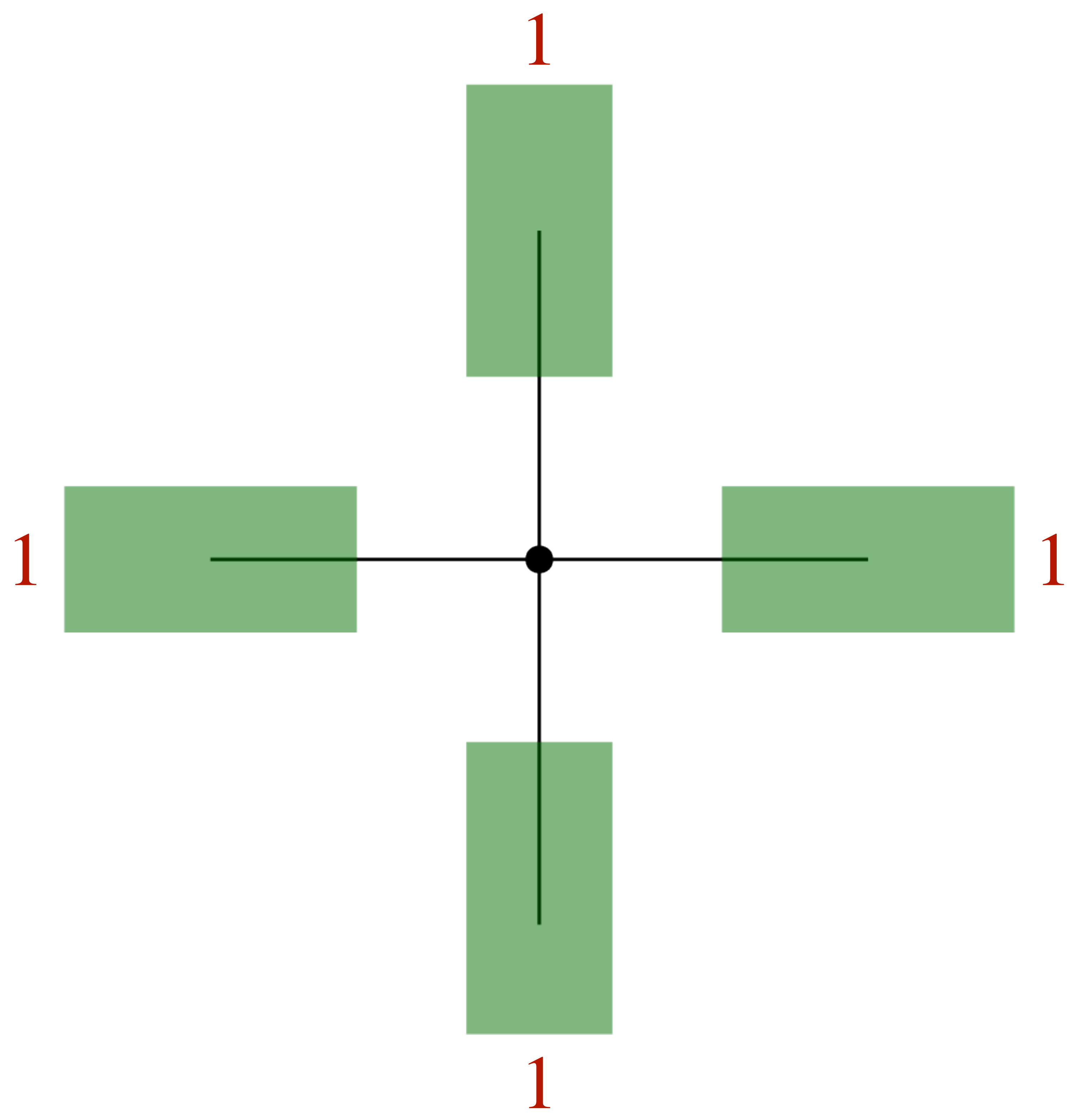

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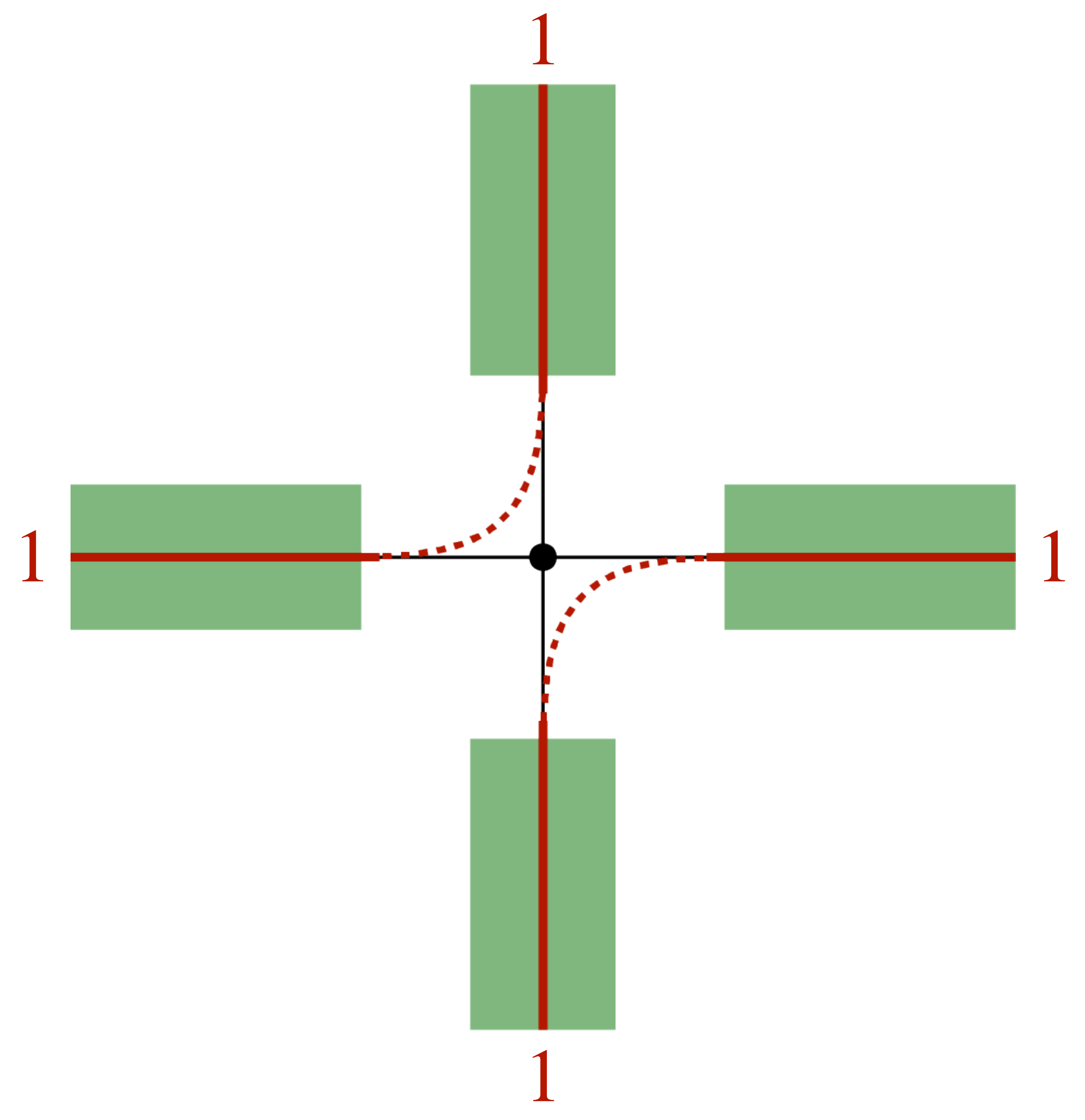

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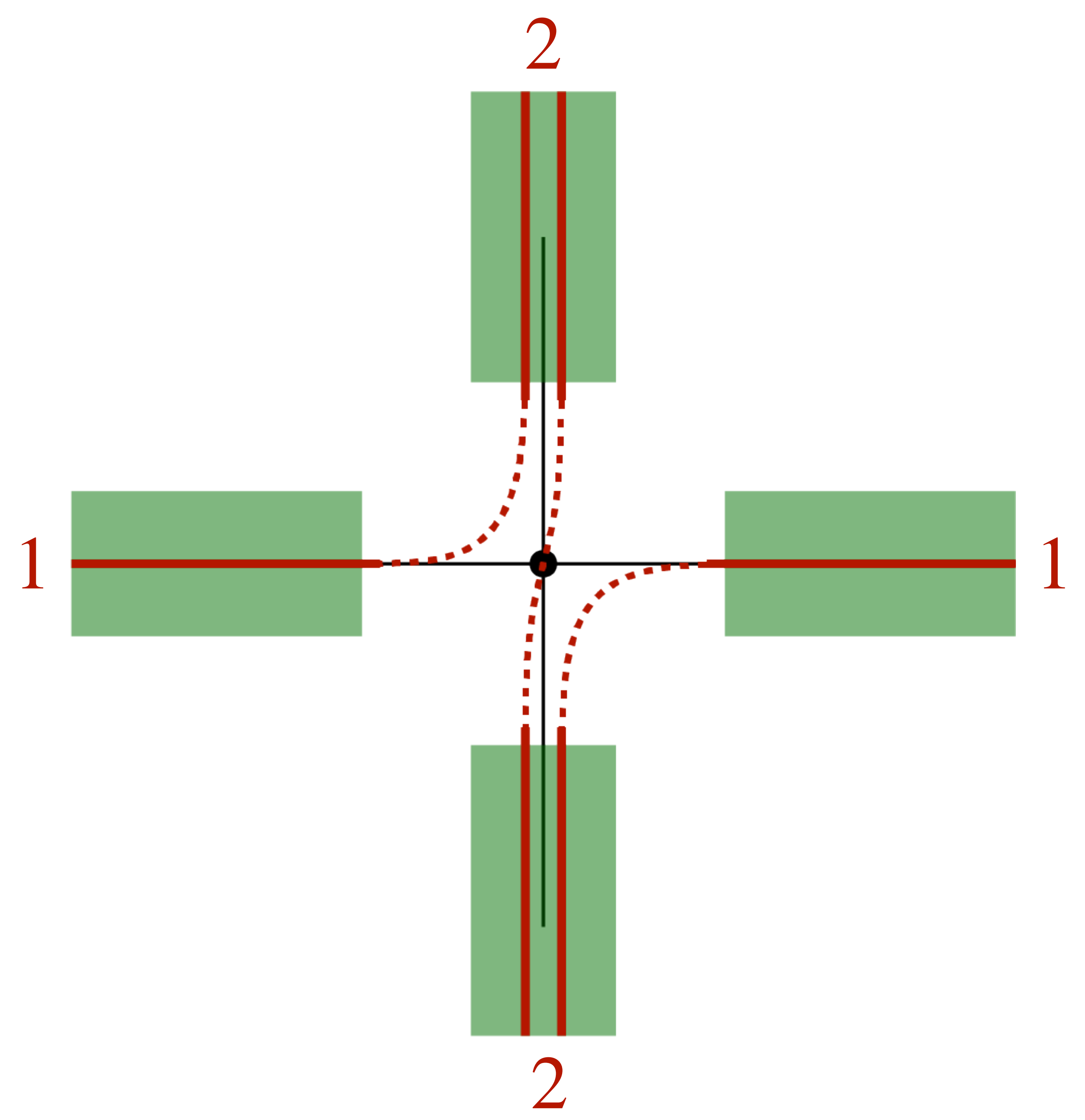

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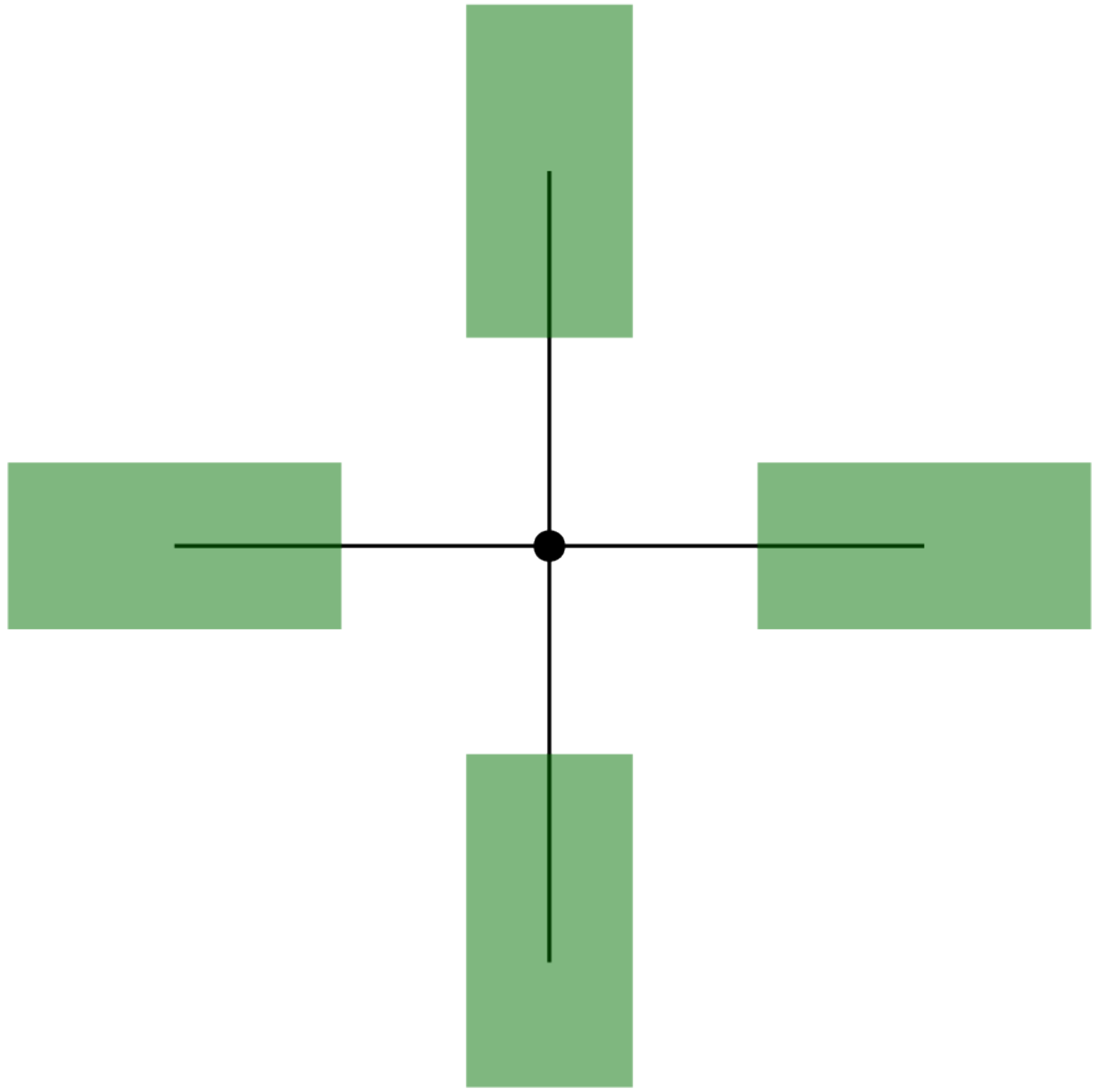

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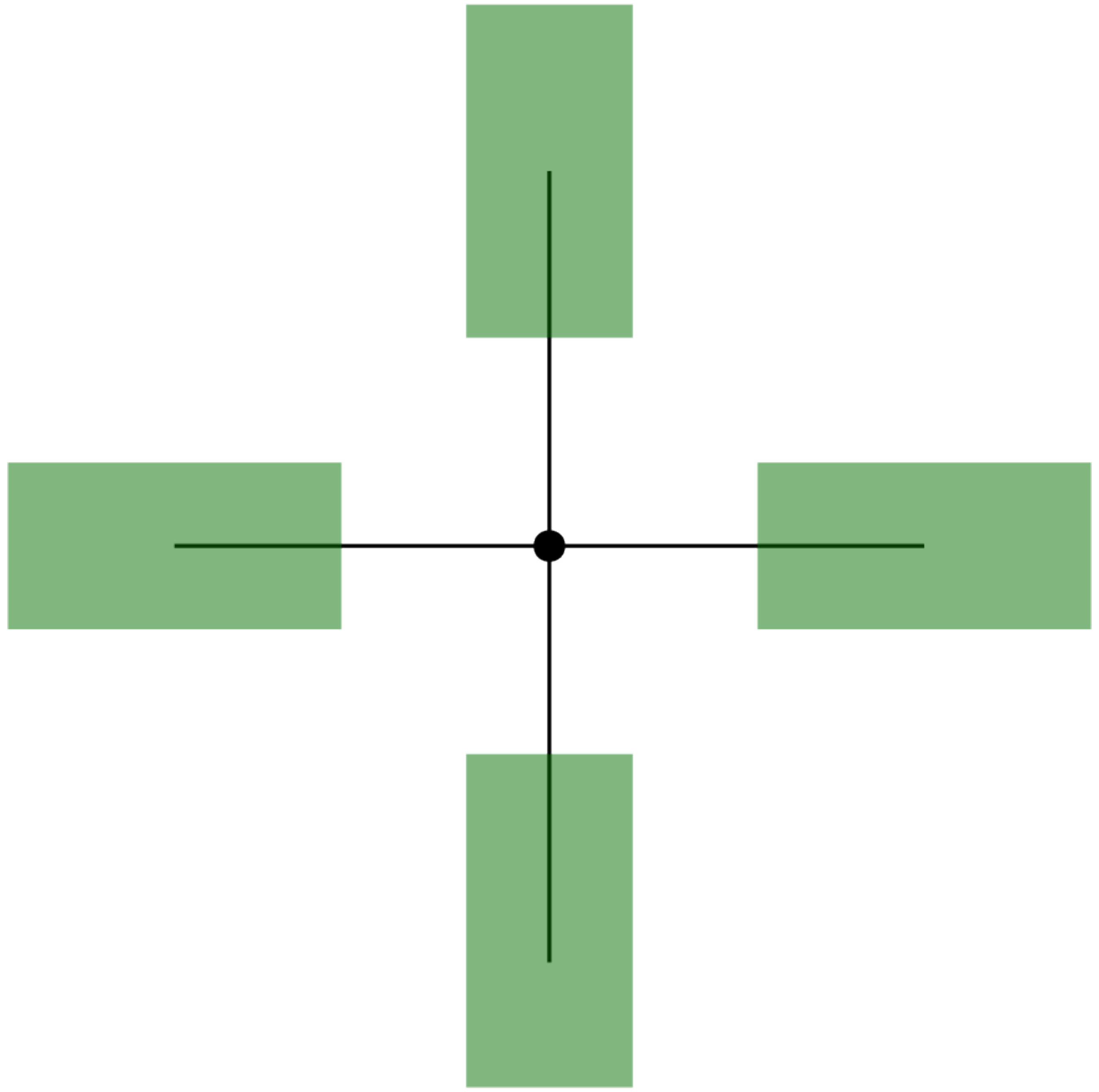

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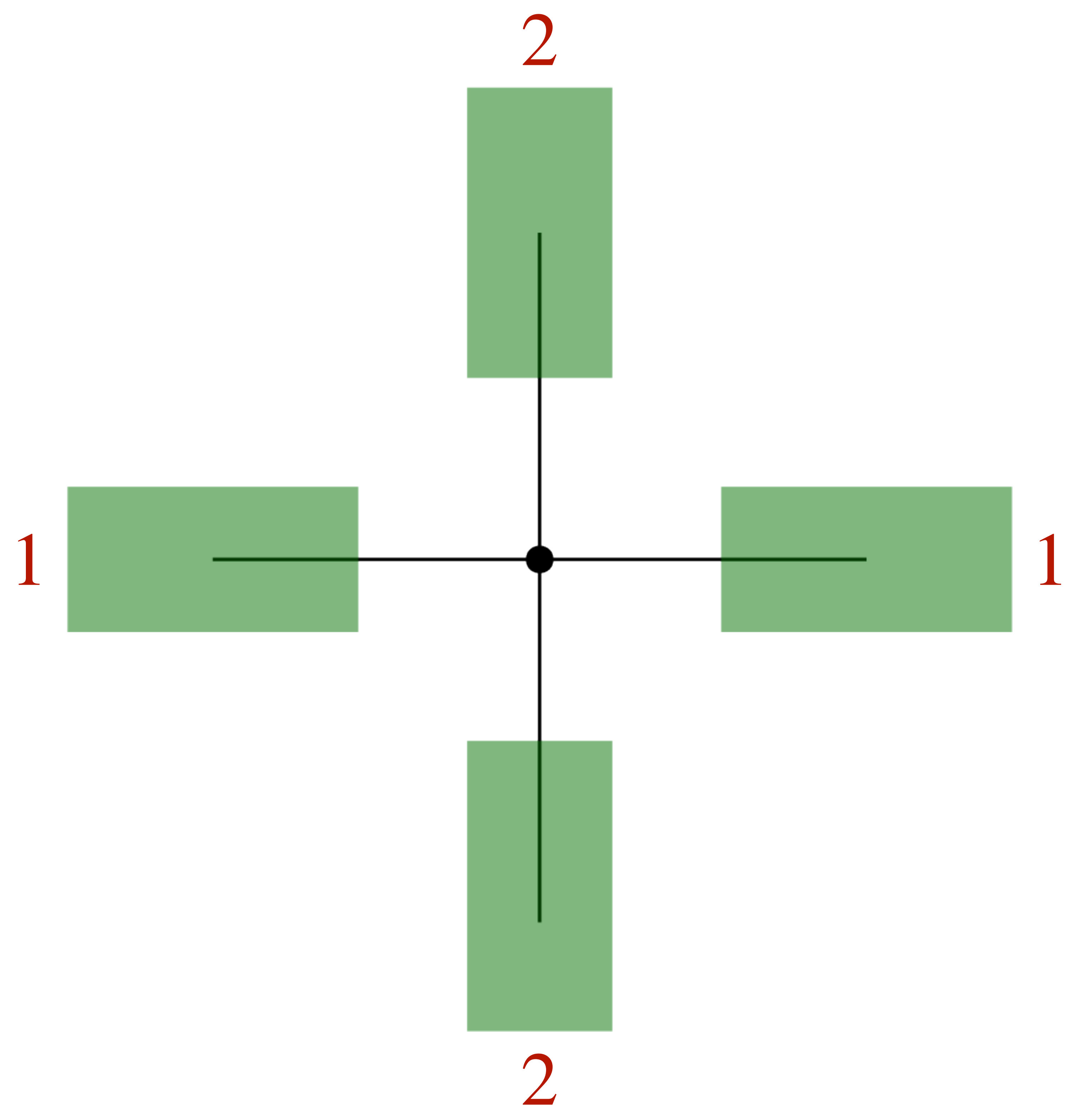

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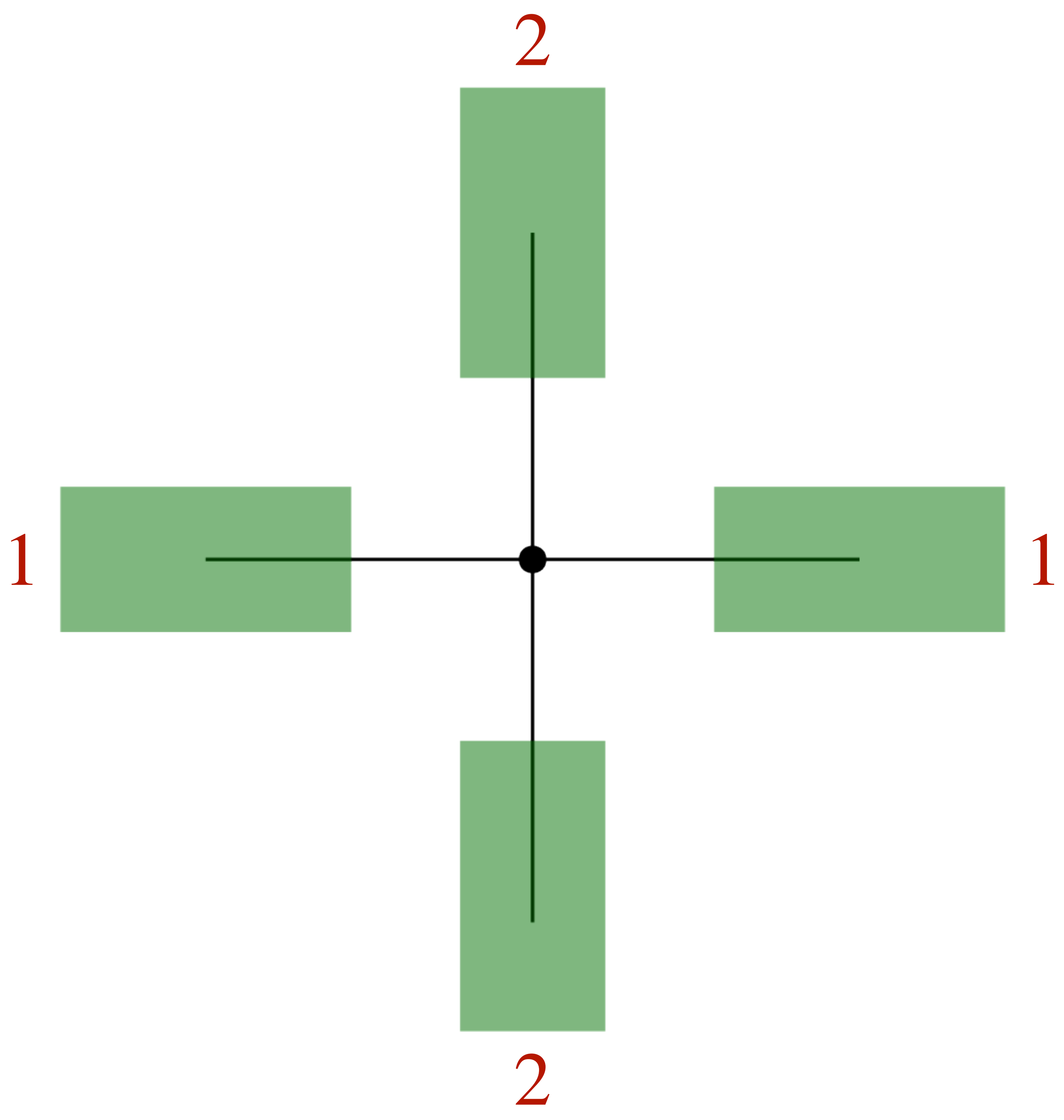

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* Refer to paper for generalization to $\geq$

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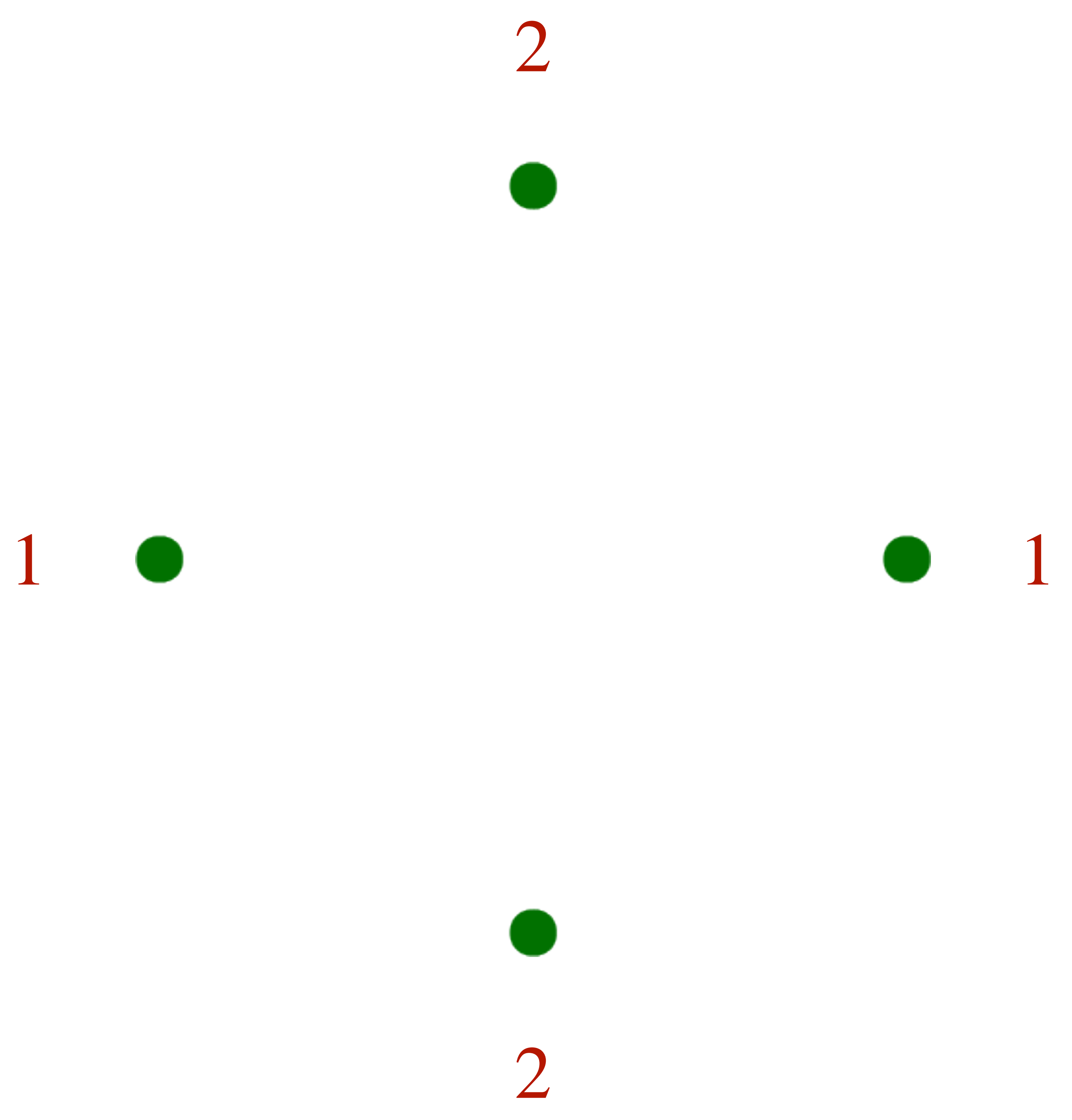

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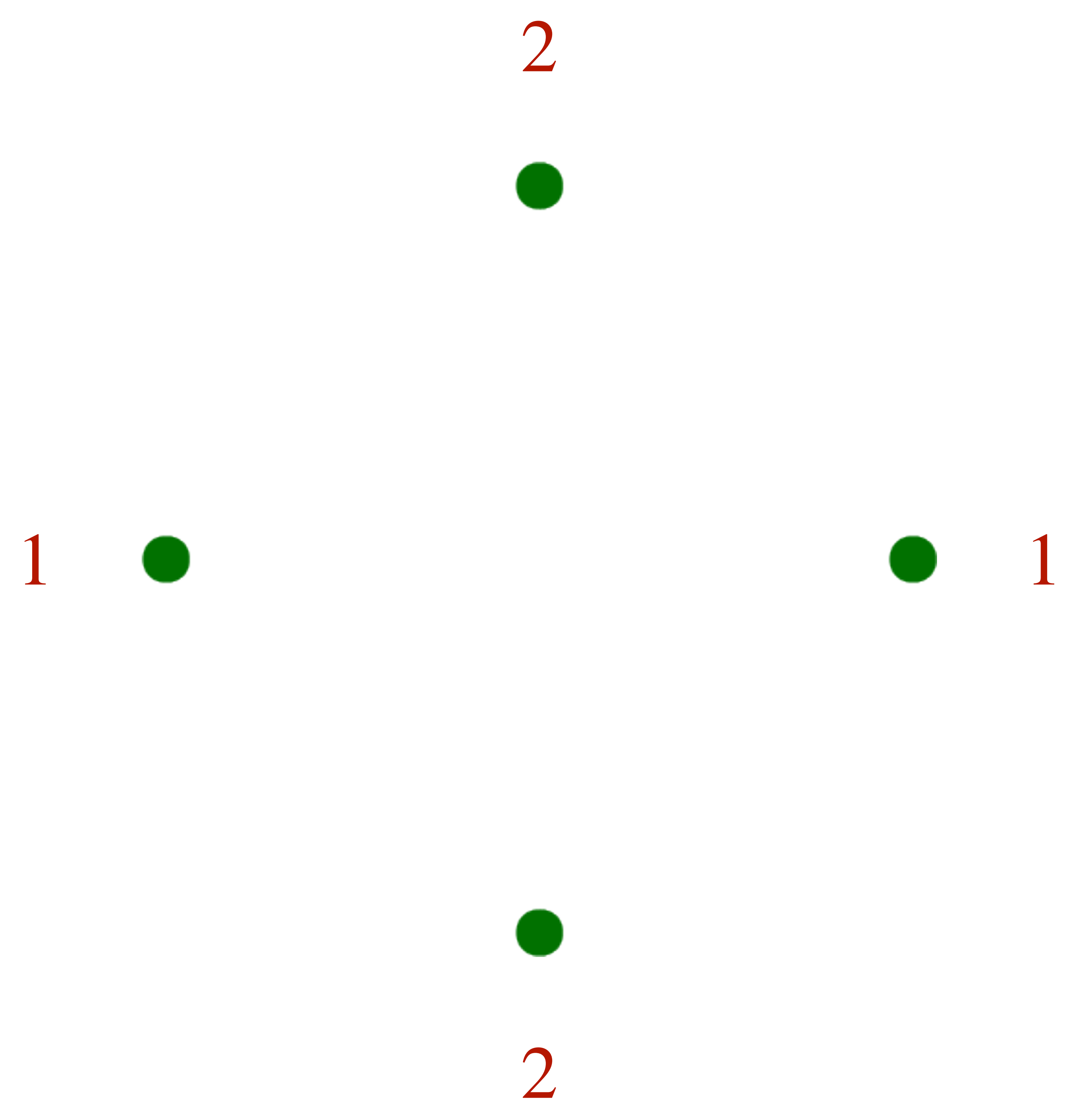
Lemma. We can construct a d-vertex tree with degrees $x_1, \ldots, x_d \geq 1$ satisfying $\sum_{i=1}^{d} x_i = 2(d-1)$ in $O(d)$ time.



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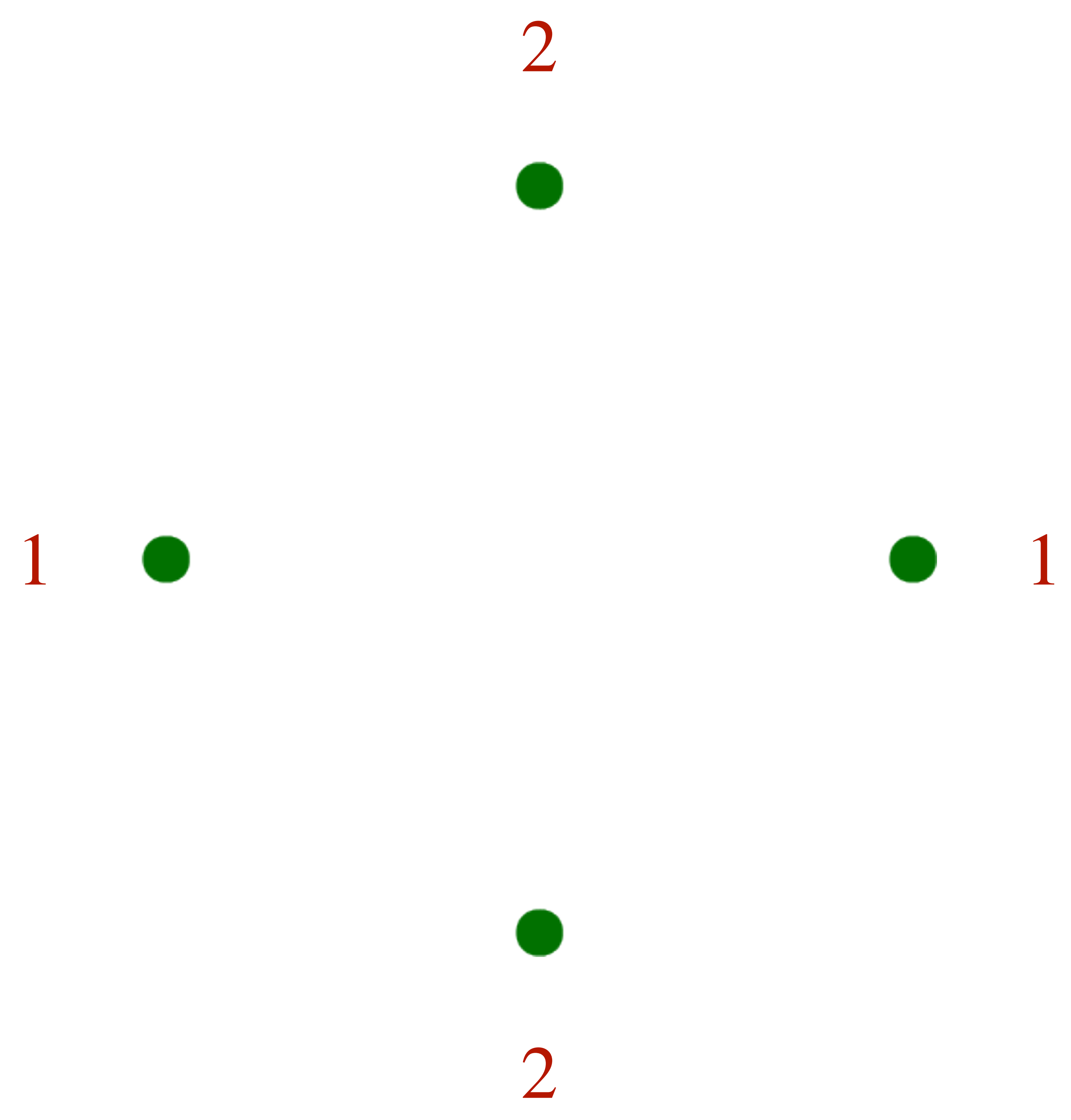

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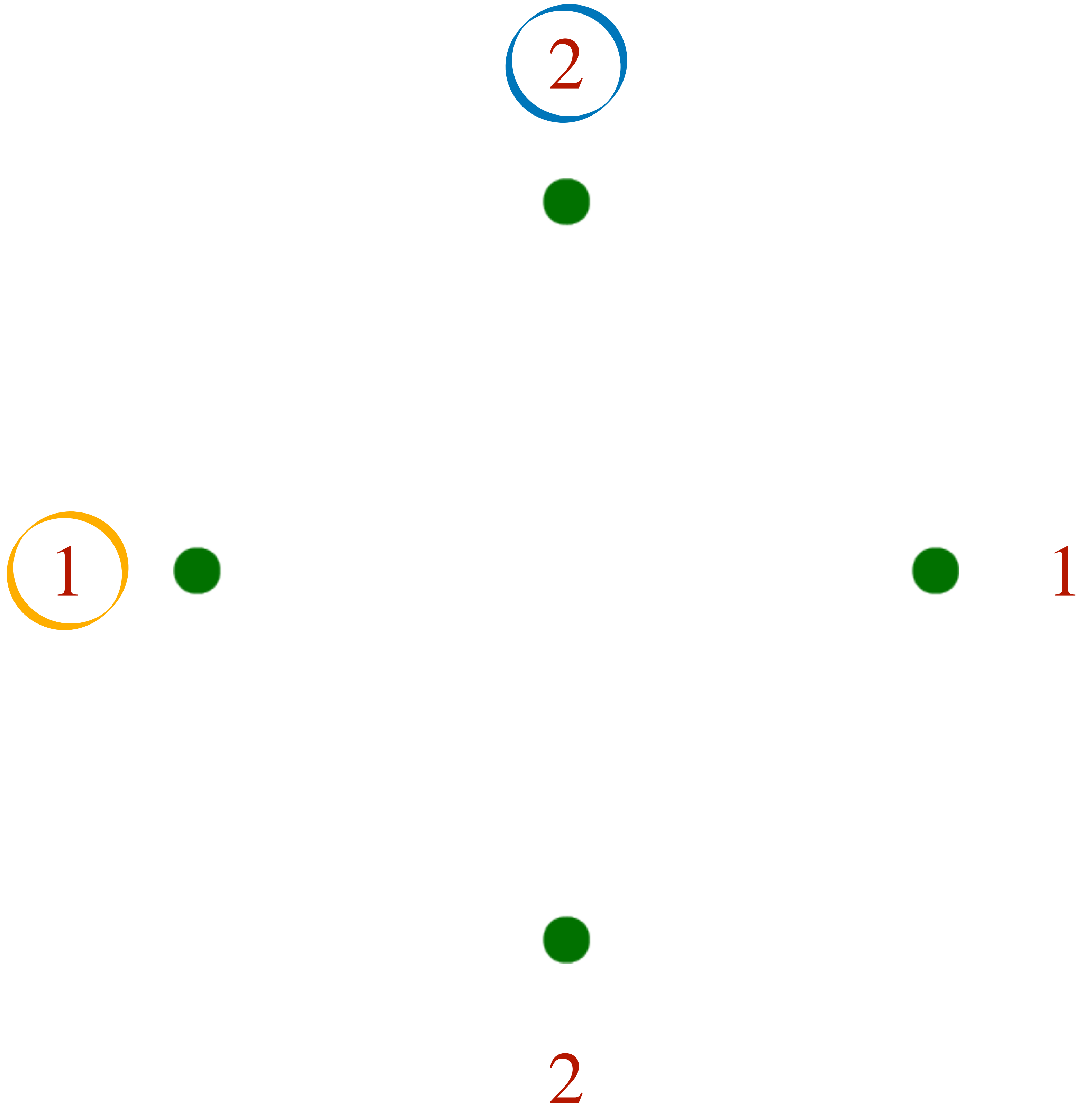

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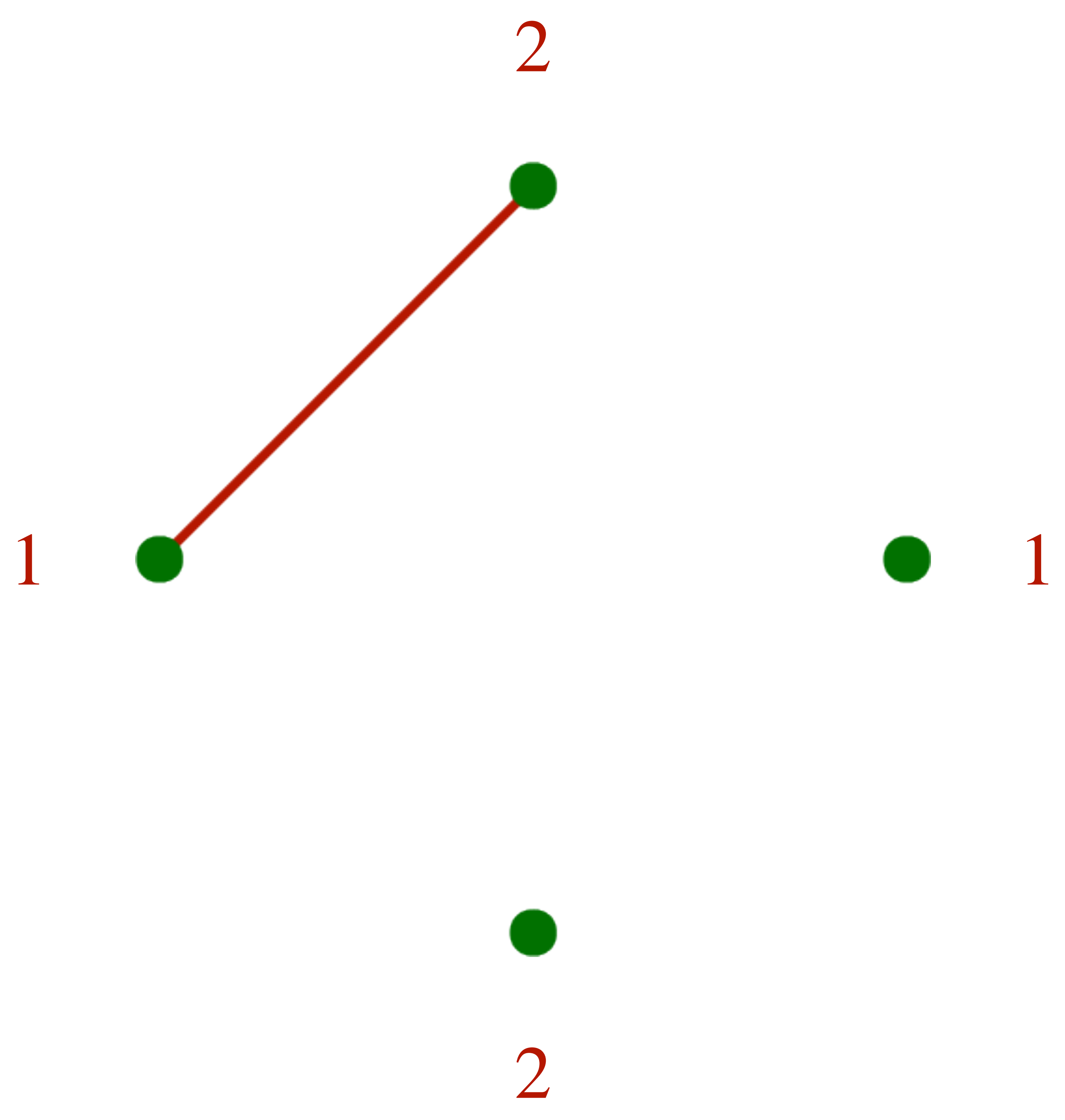

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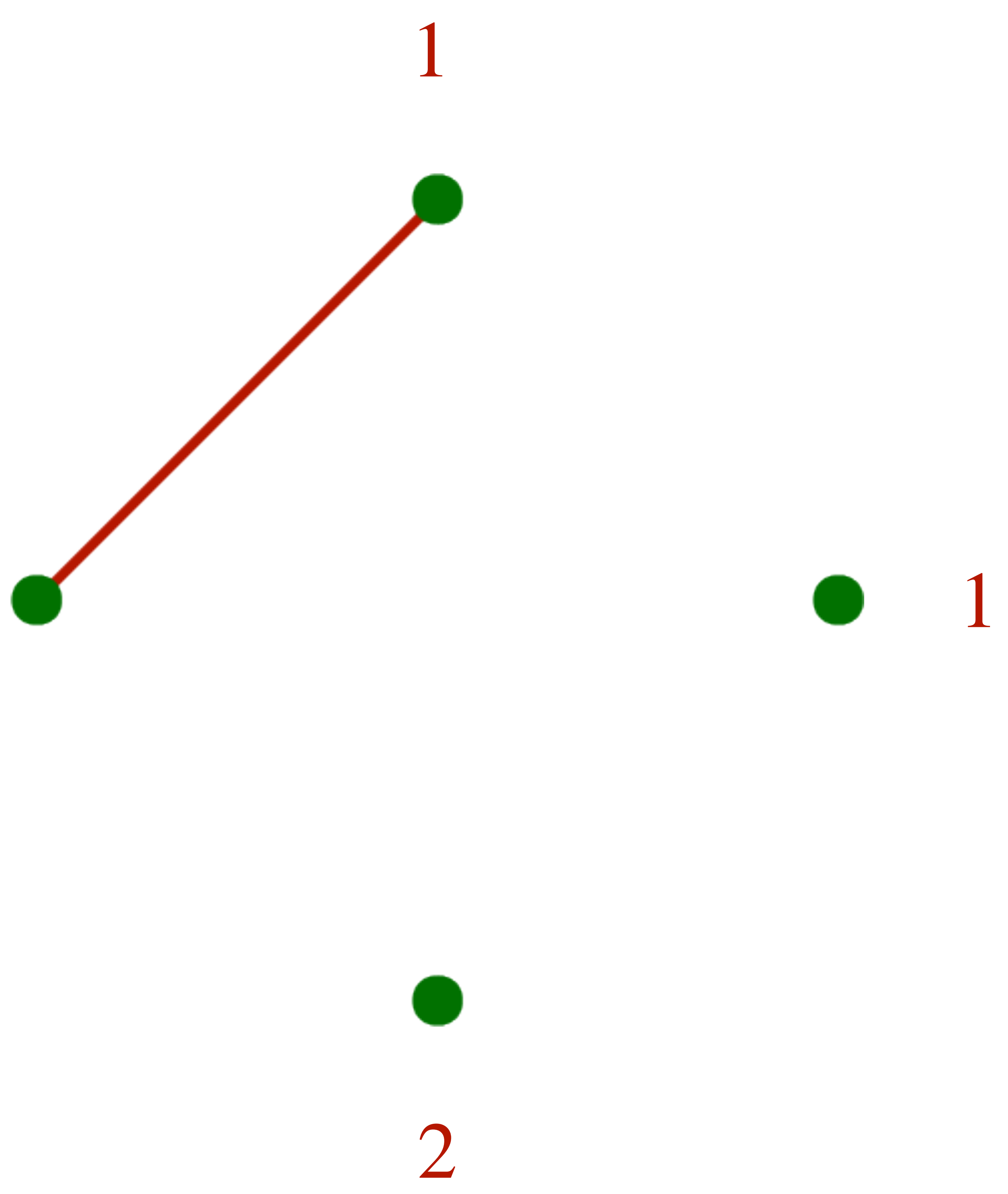

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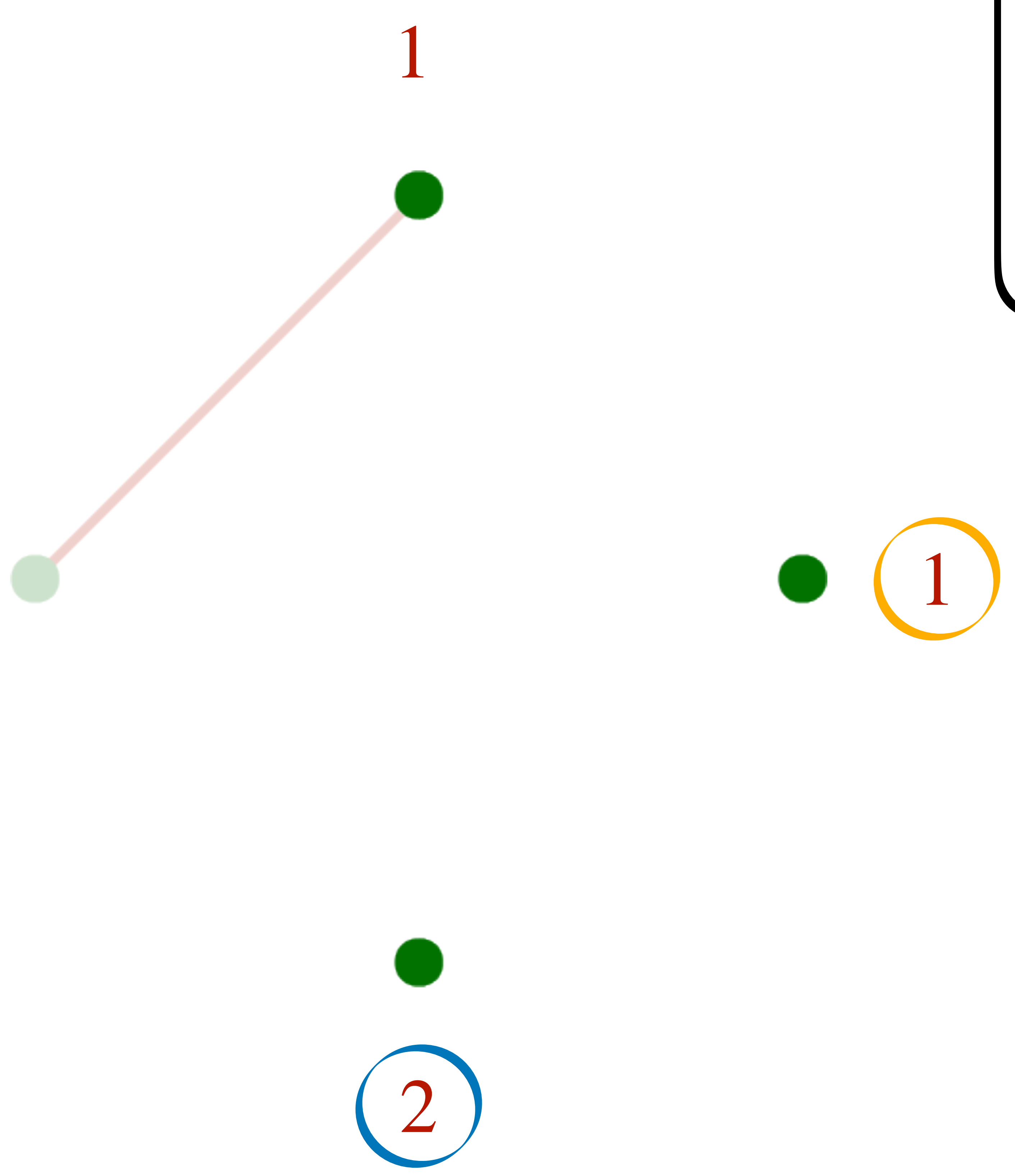

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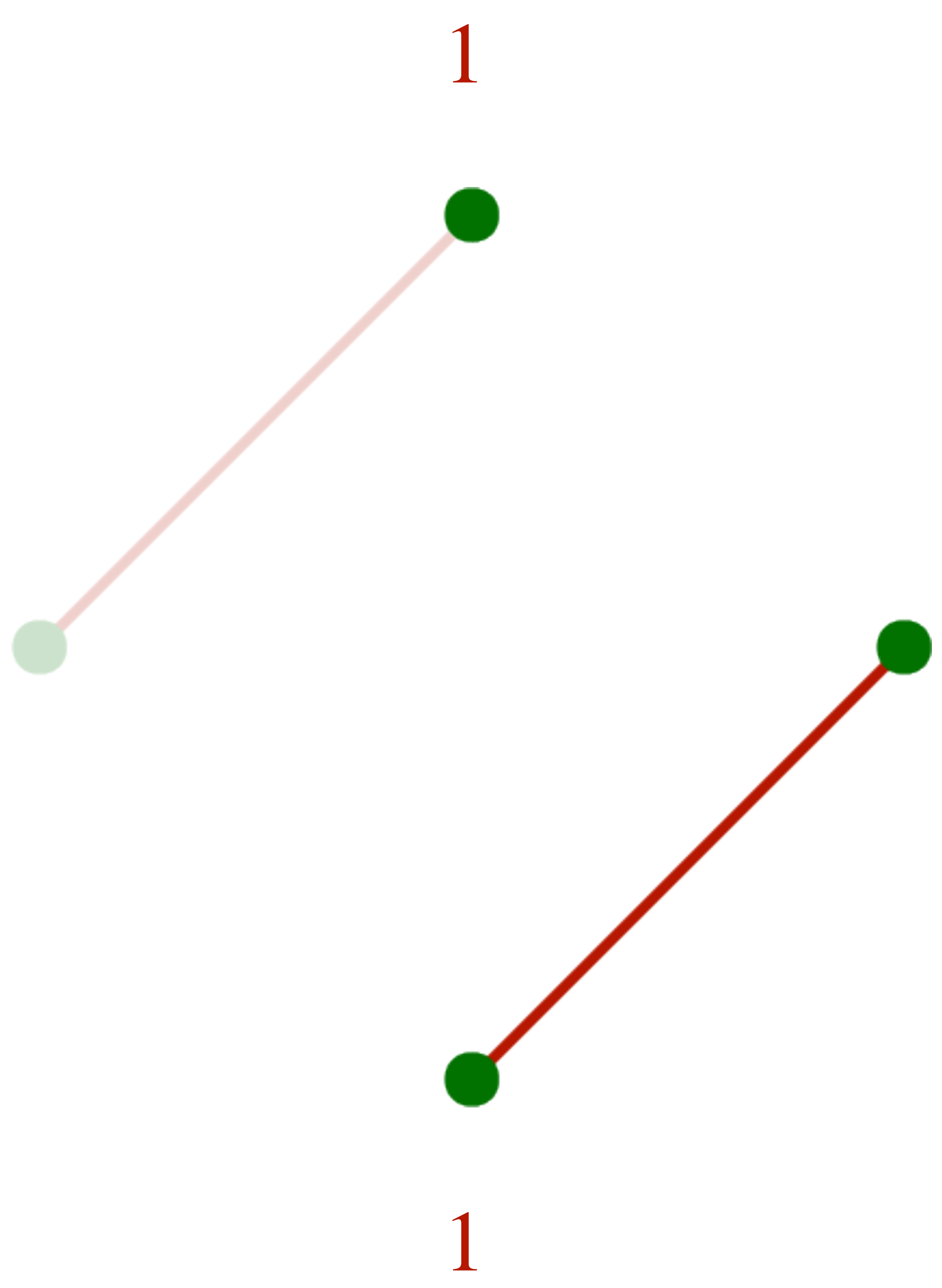

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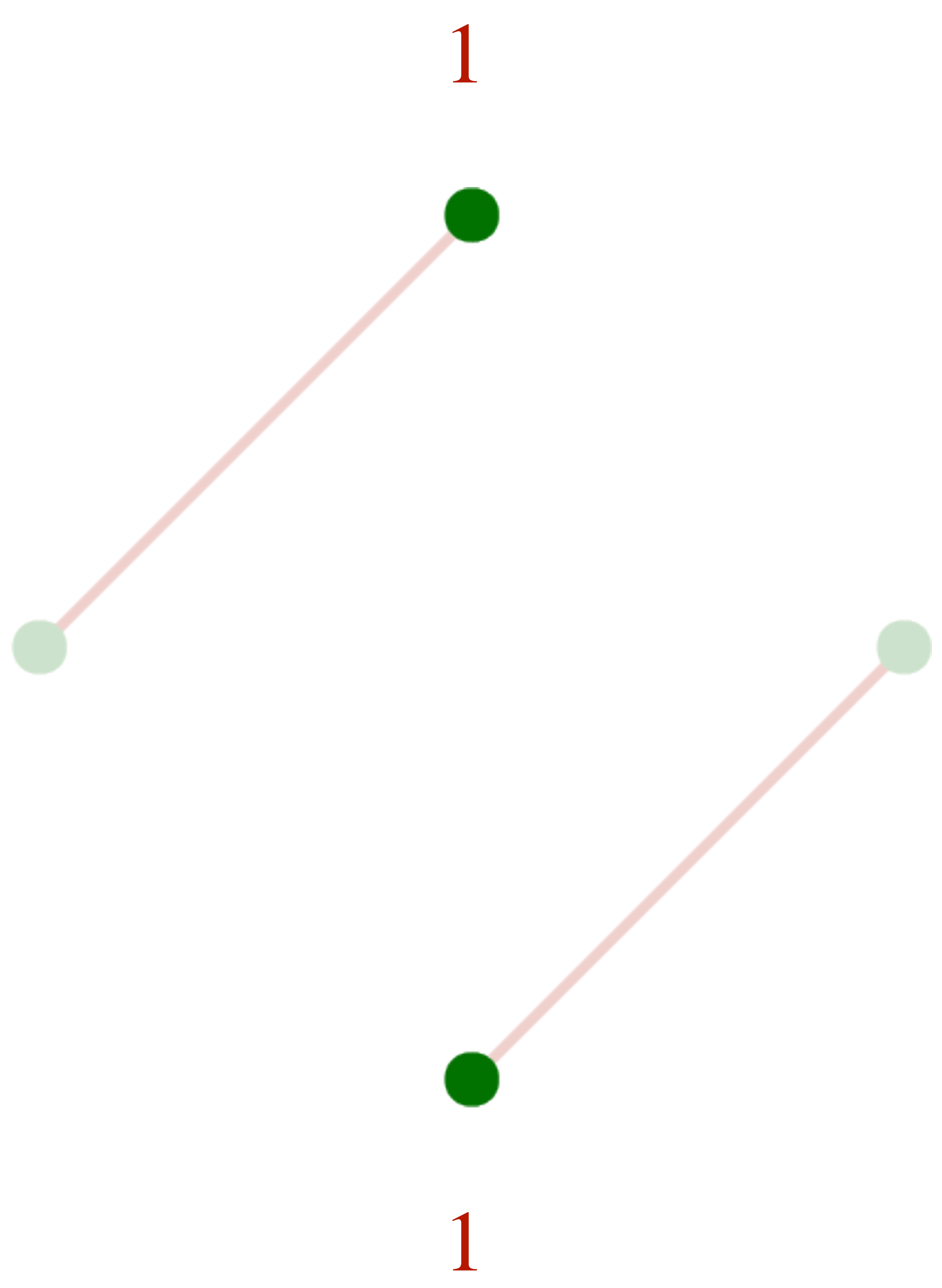

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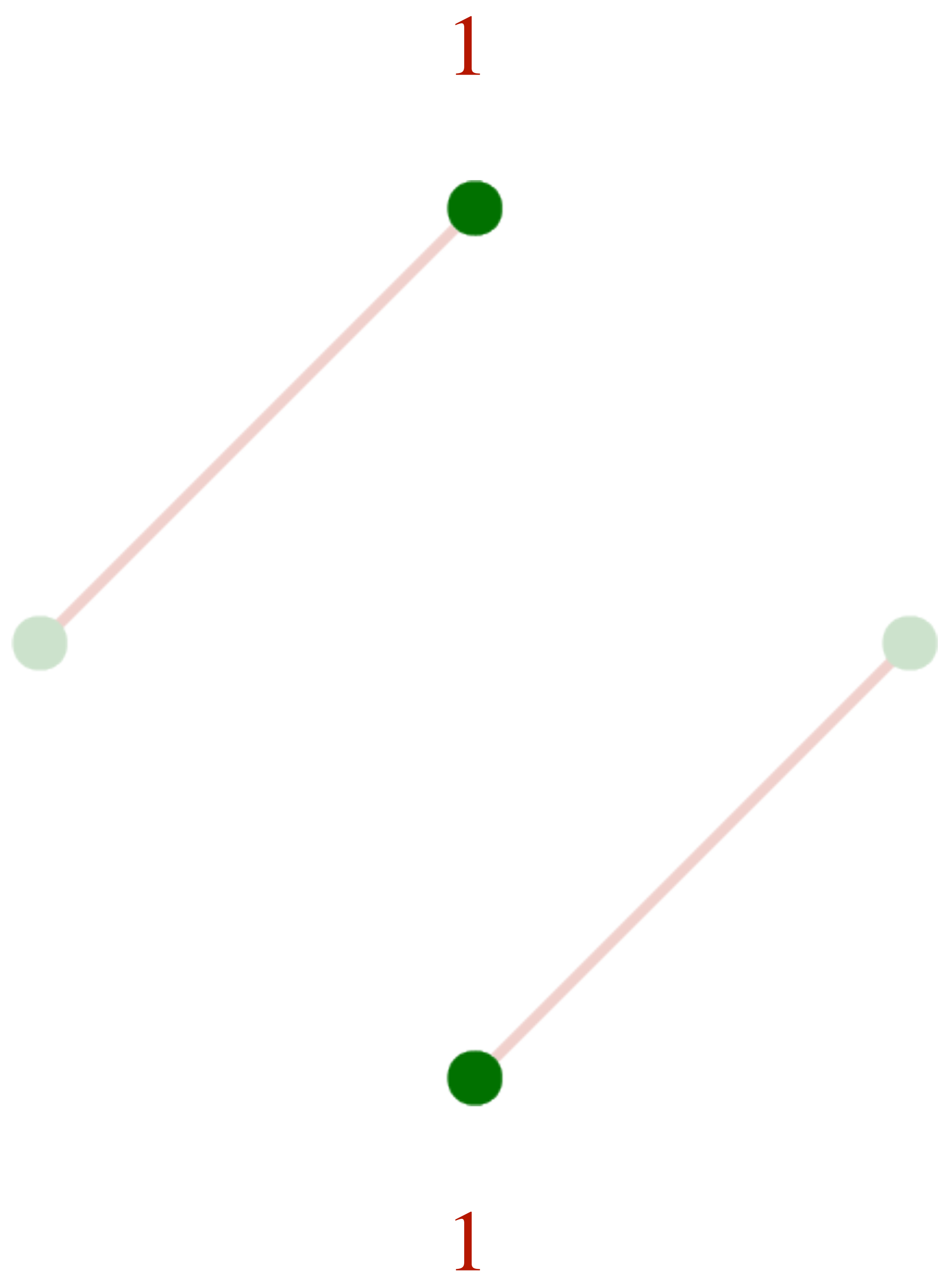

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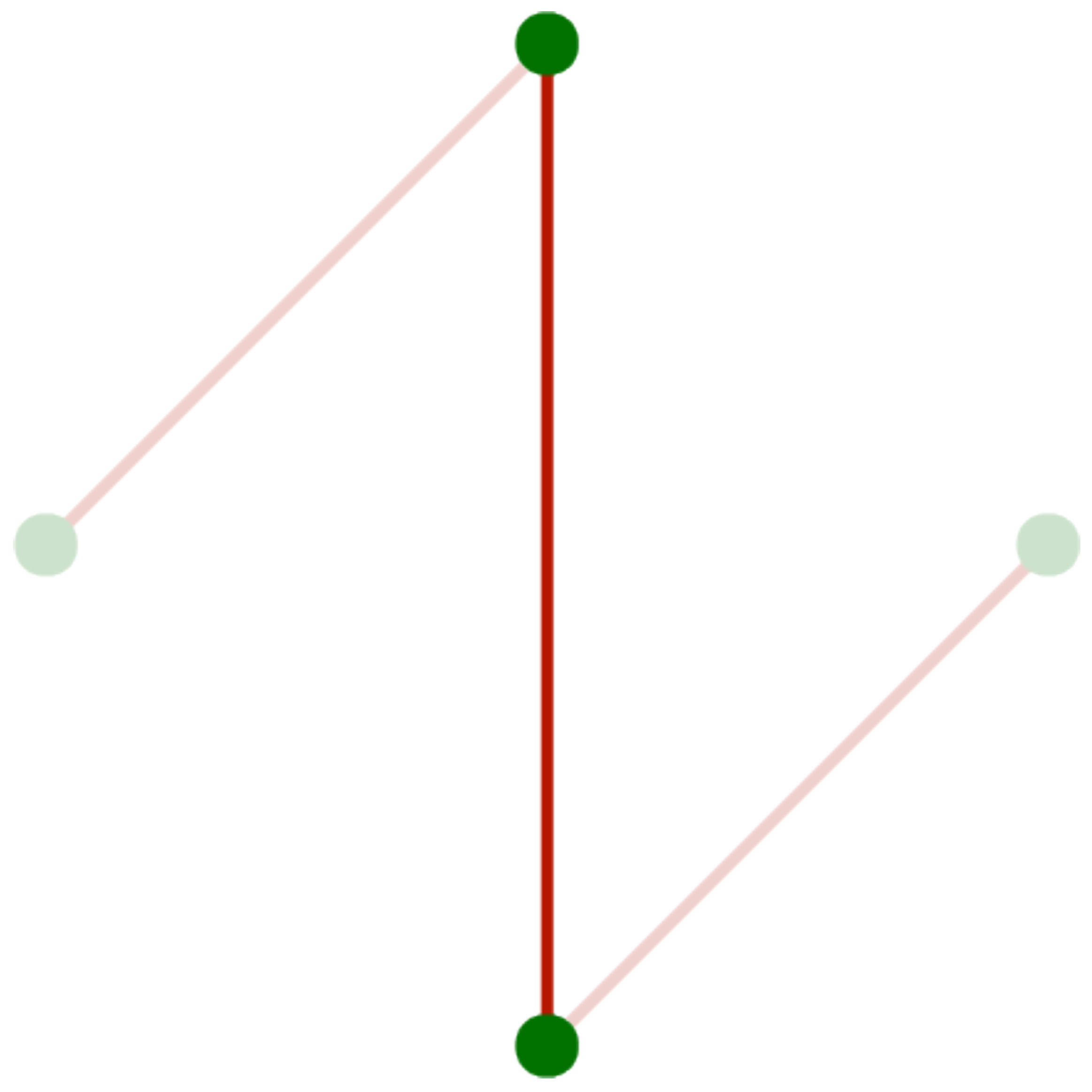

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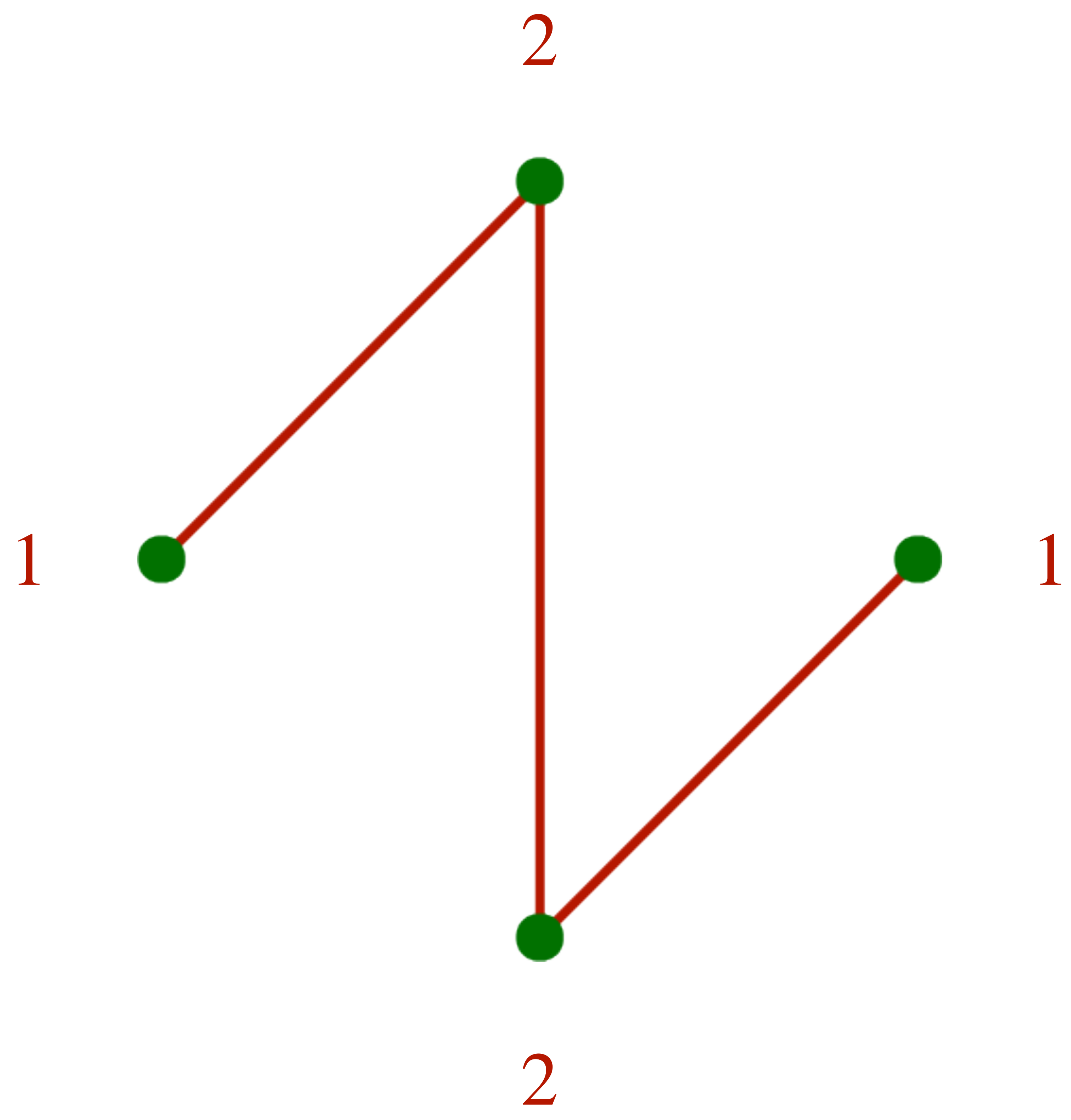

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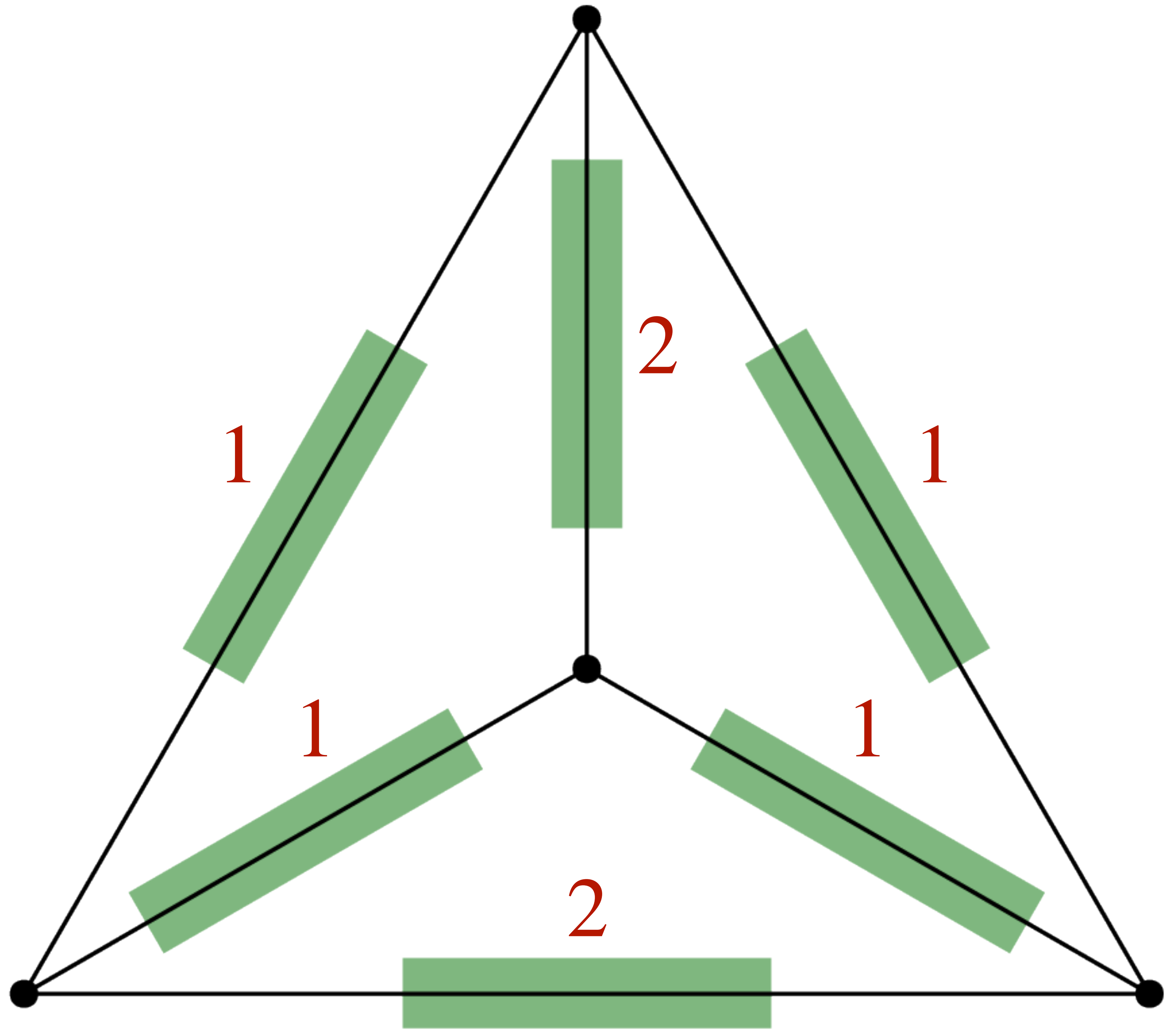

2
1
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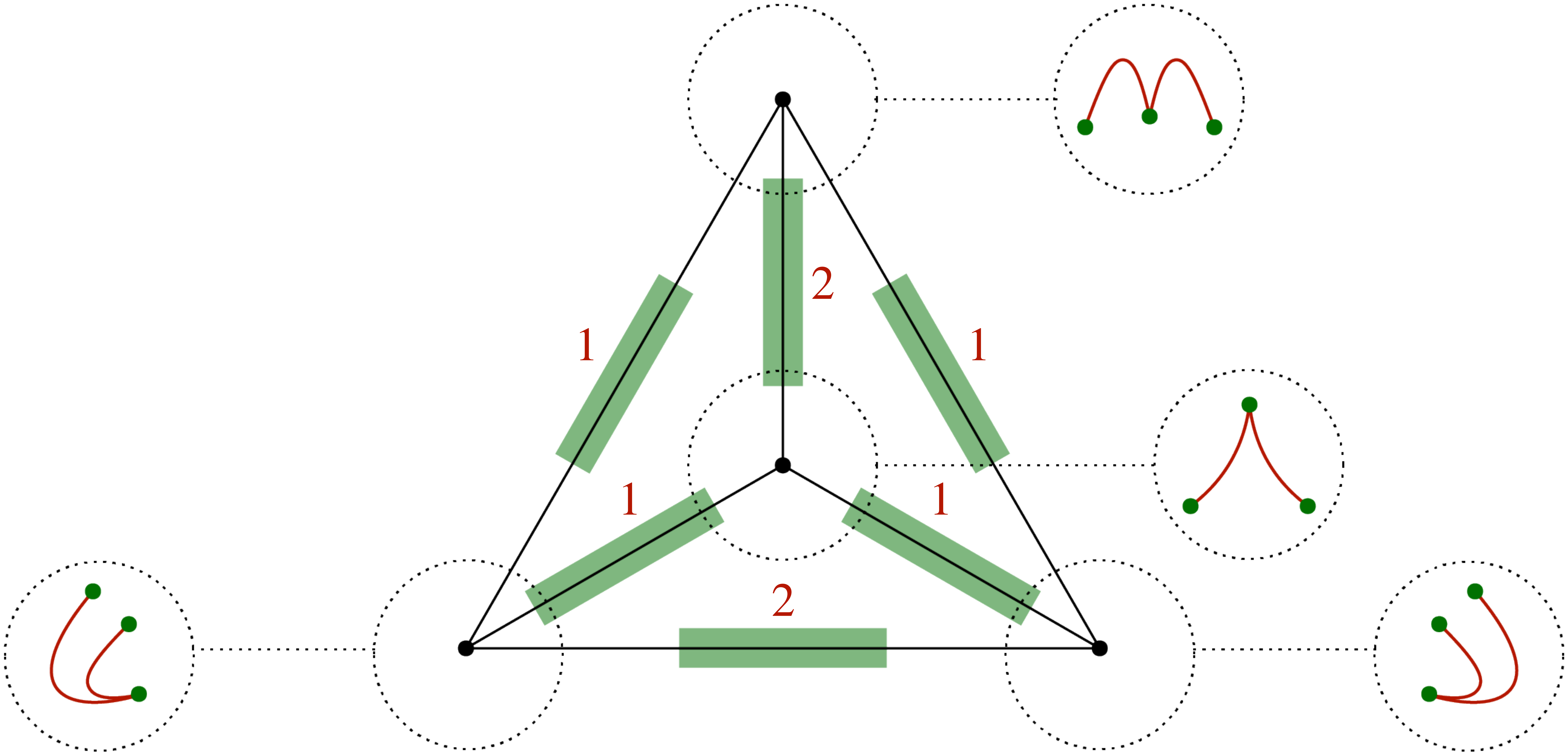
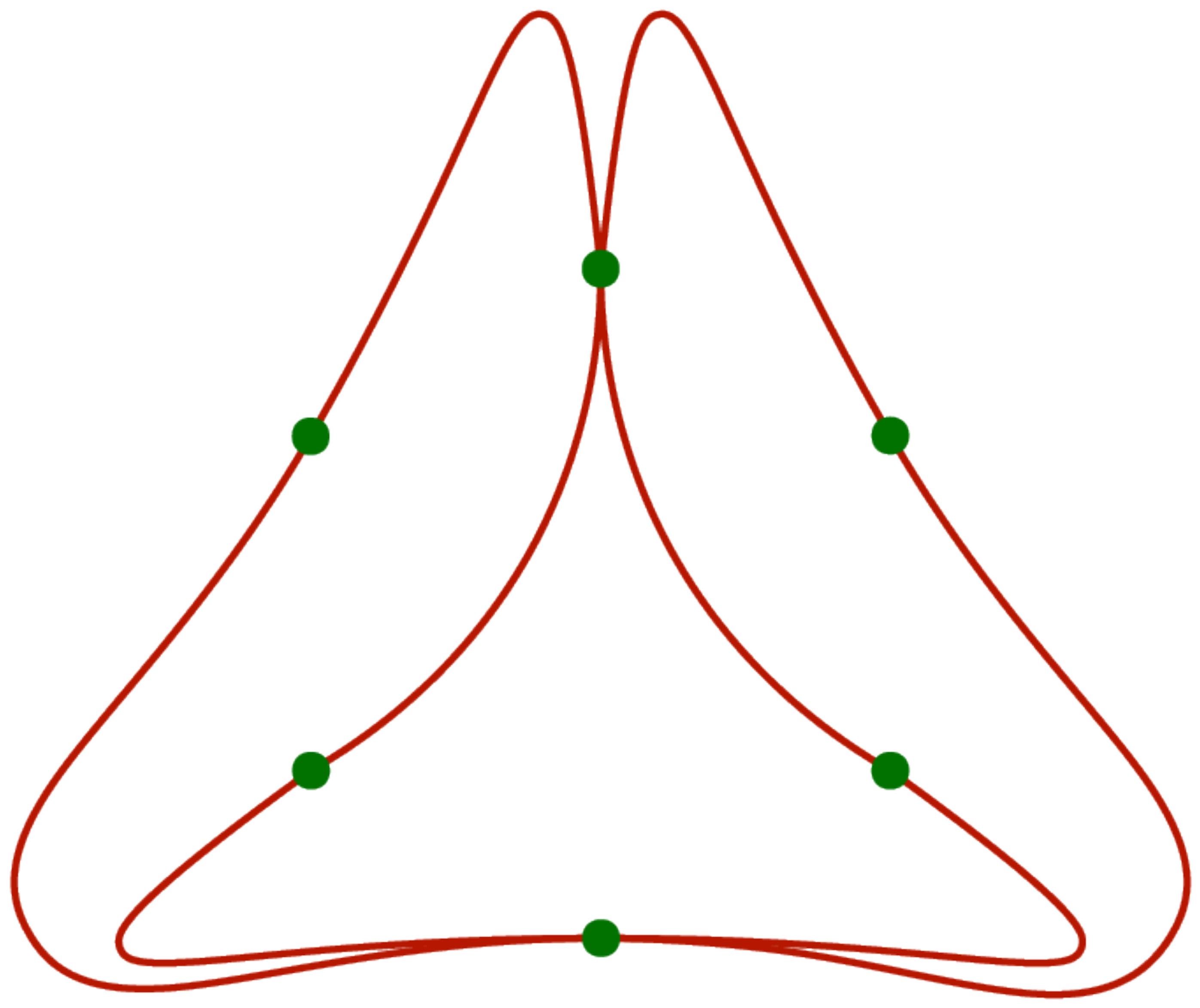
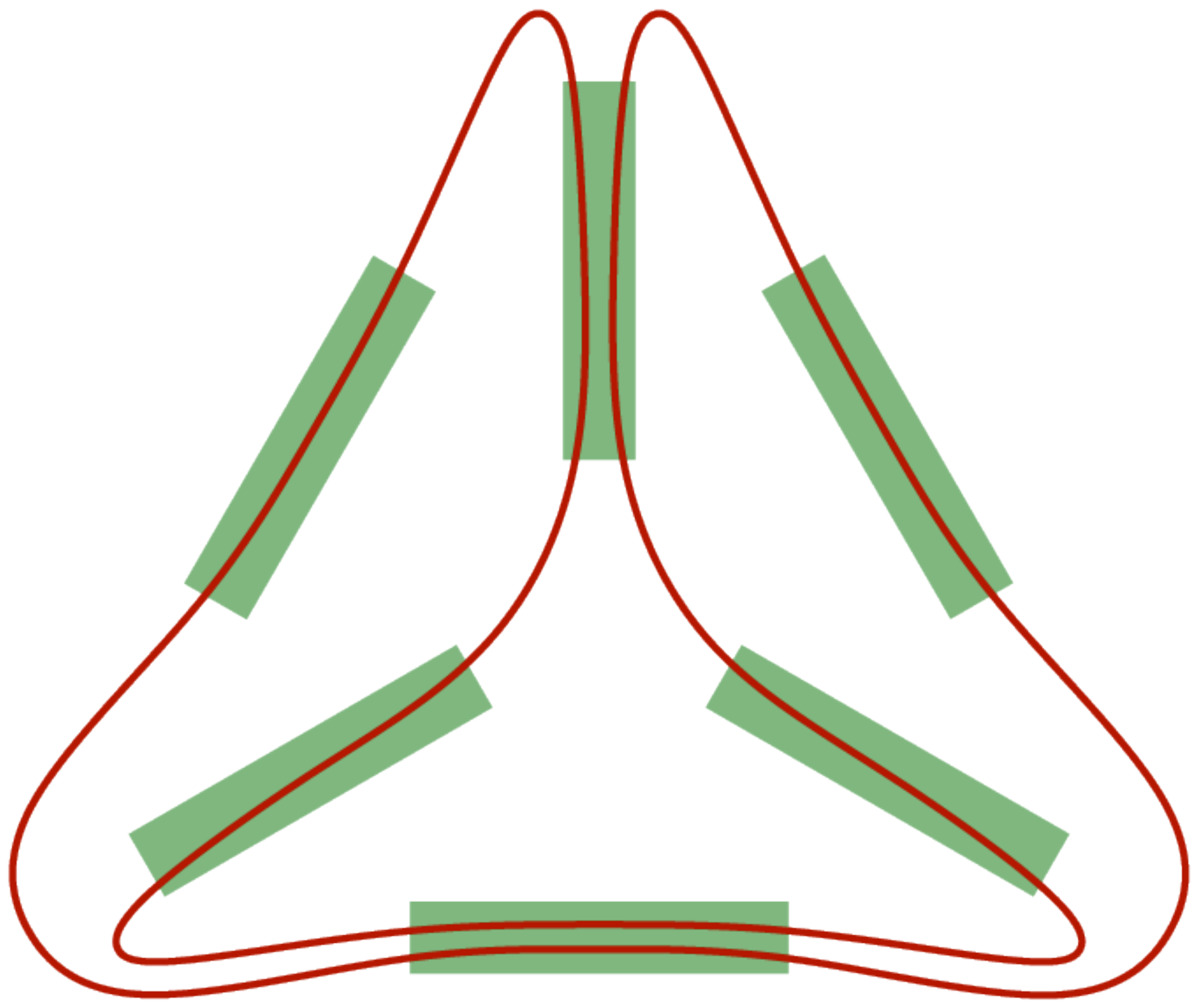
1
2


2
1
1
1
1
2

Step 2: Obtain a Threading



union of junction graphs



forbidden pattern Euler tour
[Bosboom et al. 2020]

Bounds

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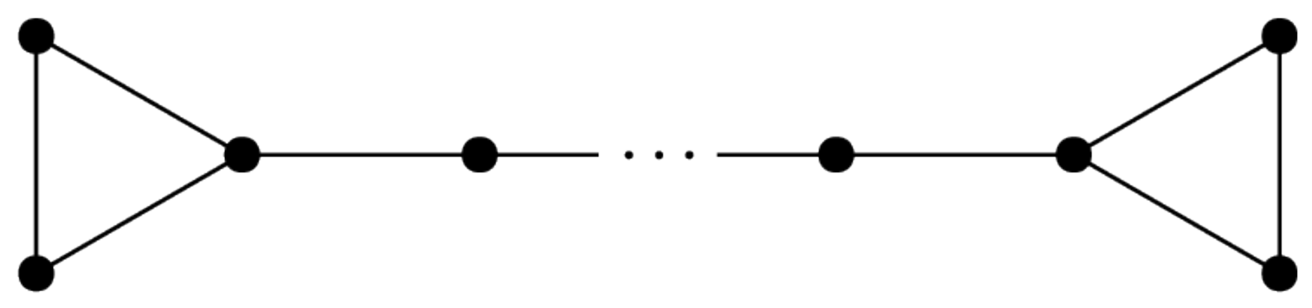

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Proof. $$\sum_{uv \in E} x_{uv} = \frac{1}{2} \sum_{v \in V} \sum_{u \in N(v)} x_{uv} \geq \frac{1}{2} \sum_{v \in V} 2(d(v) - 1) \underset{\substack{\uparrow \\ \text{handshaking}}}{=} 2m - n$$

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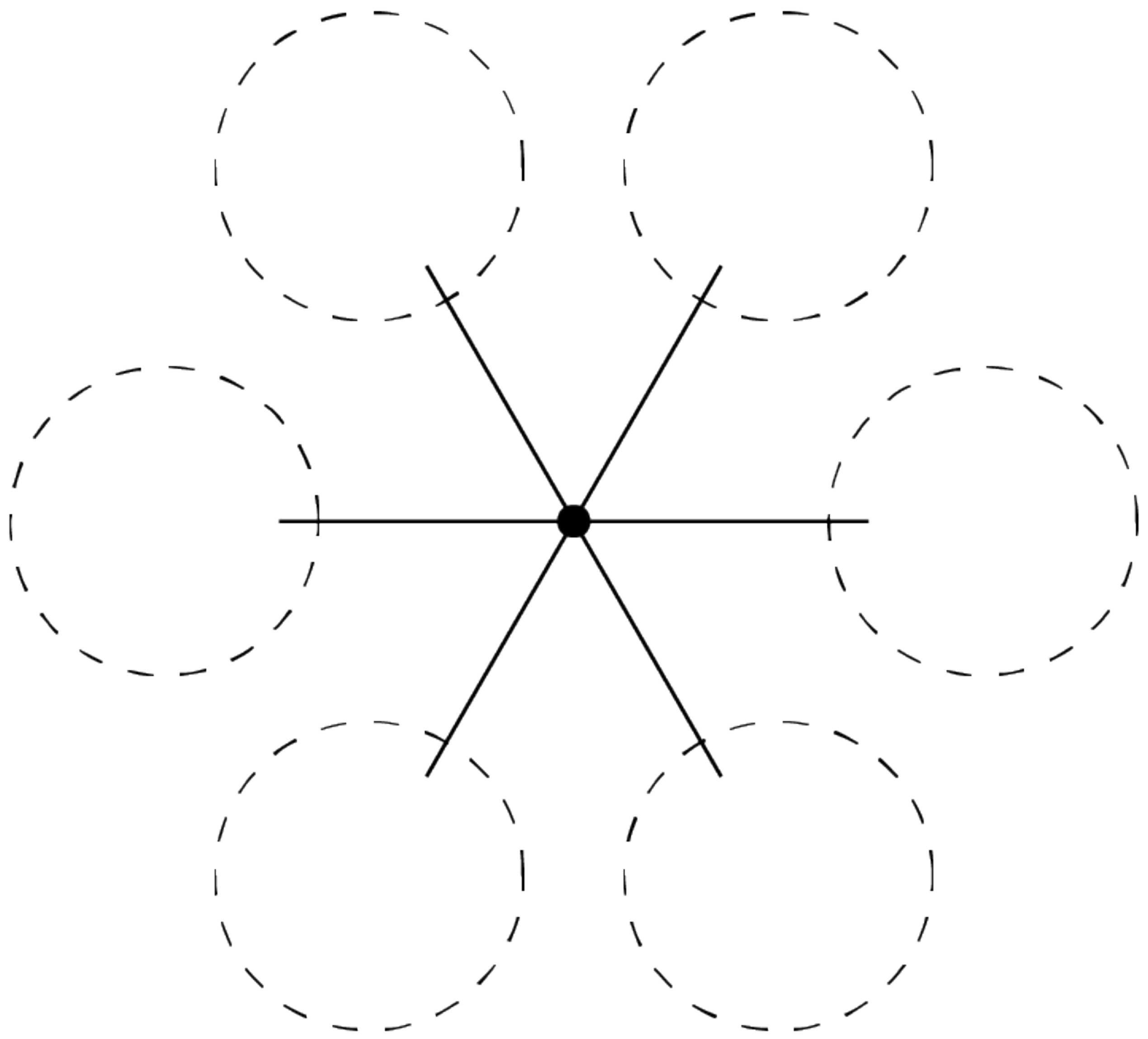

Finding a Local Threading via Perfect Matching

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For a perfect threading*, (C4) holds with an equality:

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* Refer to paper for generalization to graphs *without* perfect threadings

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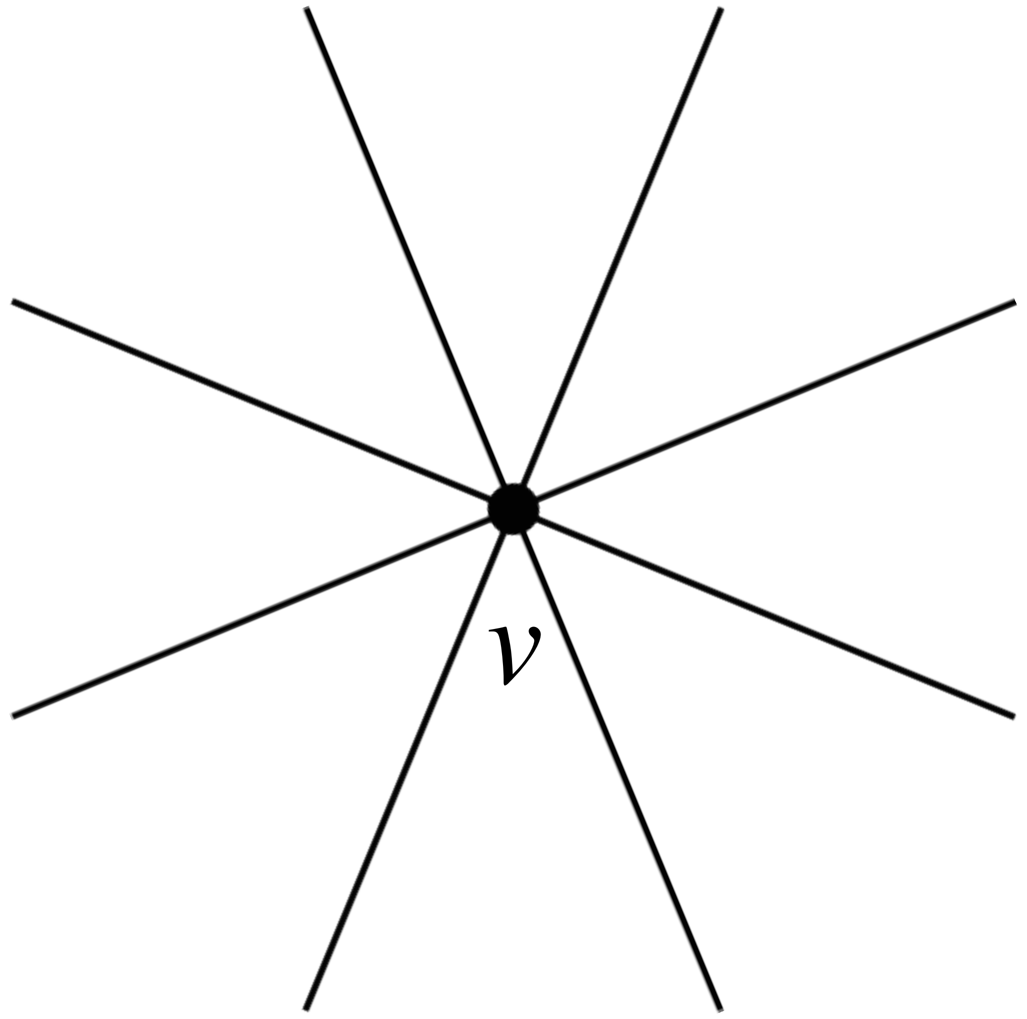

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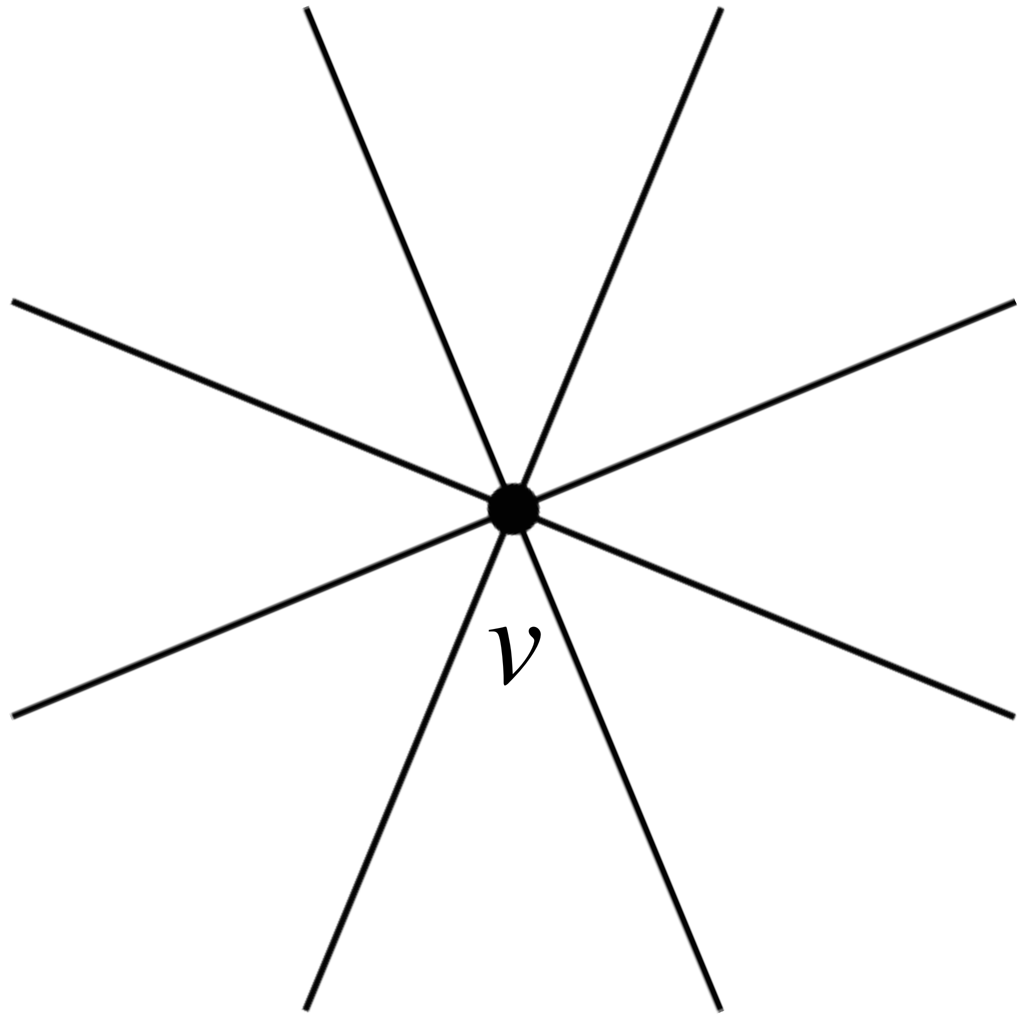

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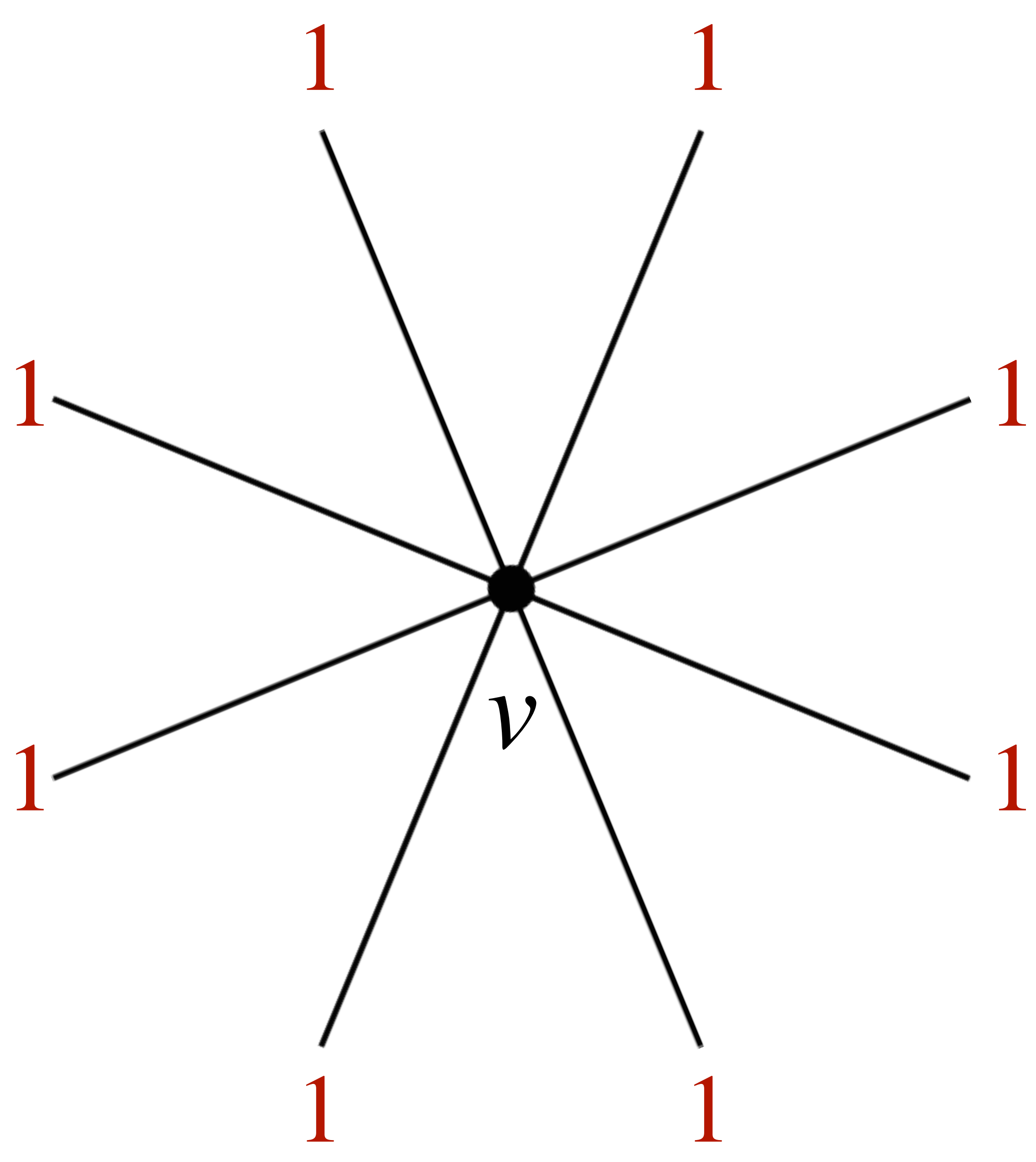

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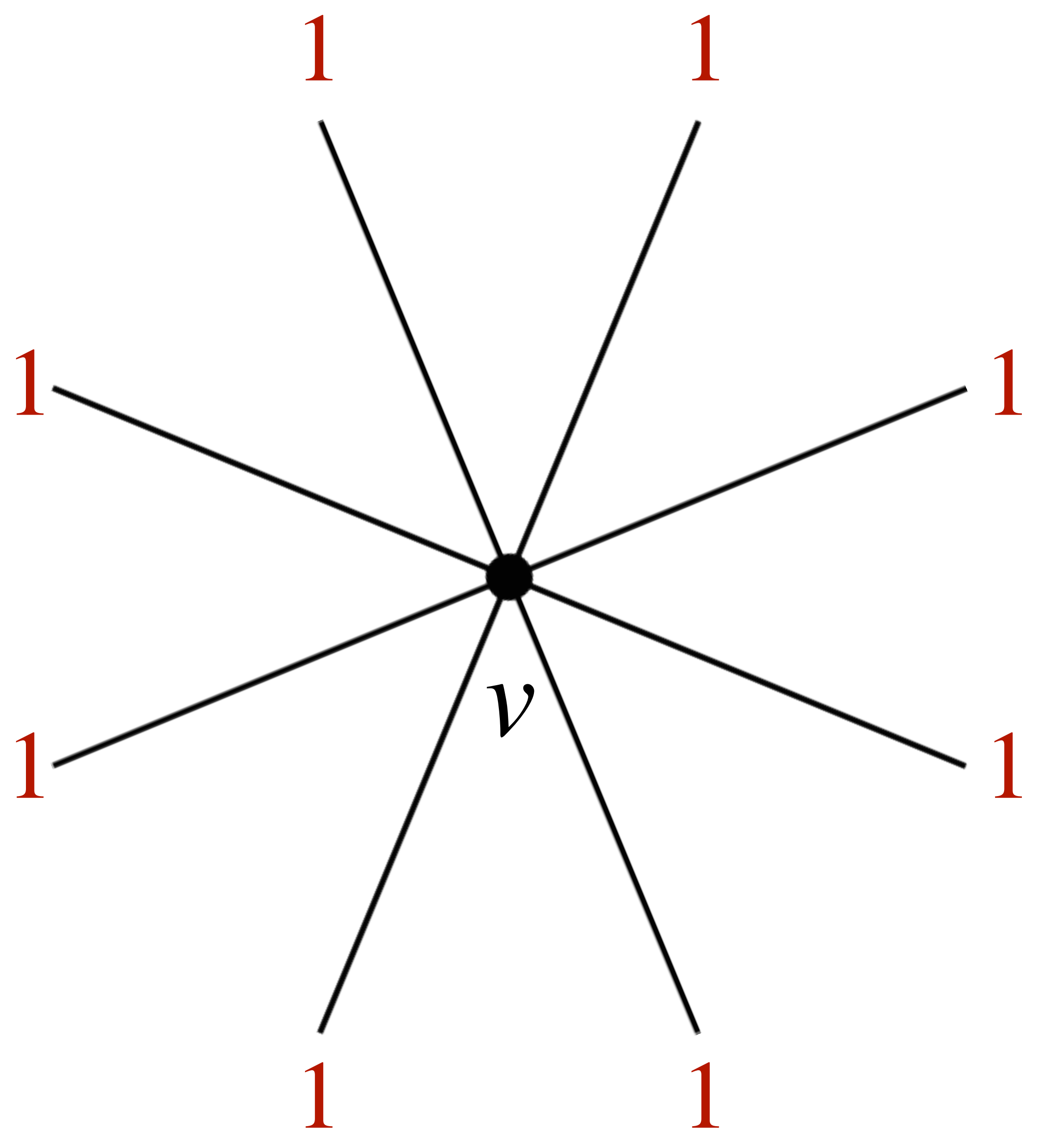

How do we distribute the *remaining* $d(v) - 2$ units amongst the edges incident to v?

Approach: Construct a graph H that has a perfect matching if and only if, for every vertex v, we can distribute $d(v) - 2$ units among its incident edges.

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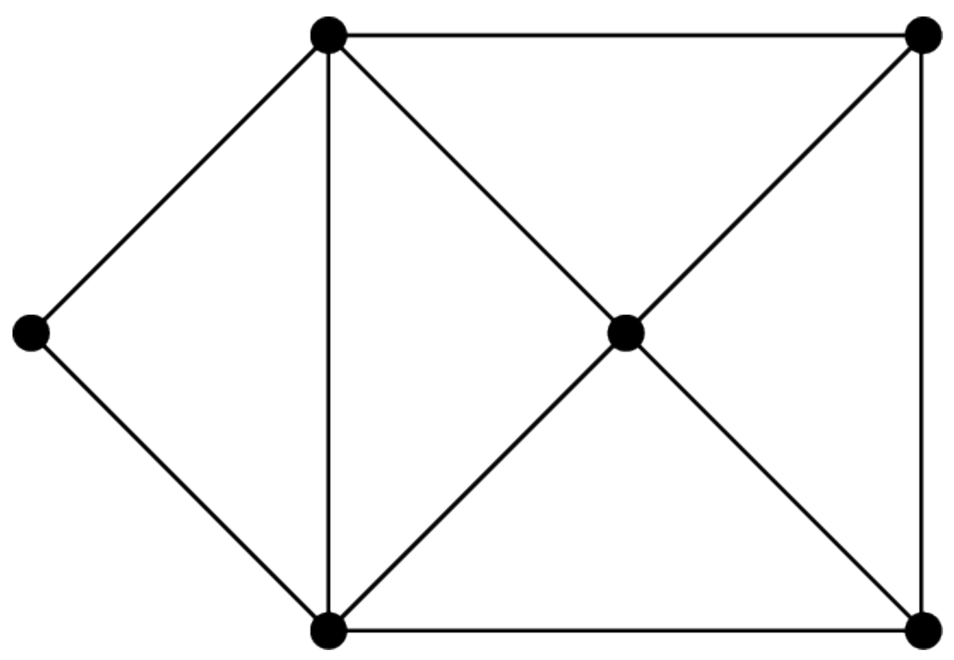
G



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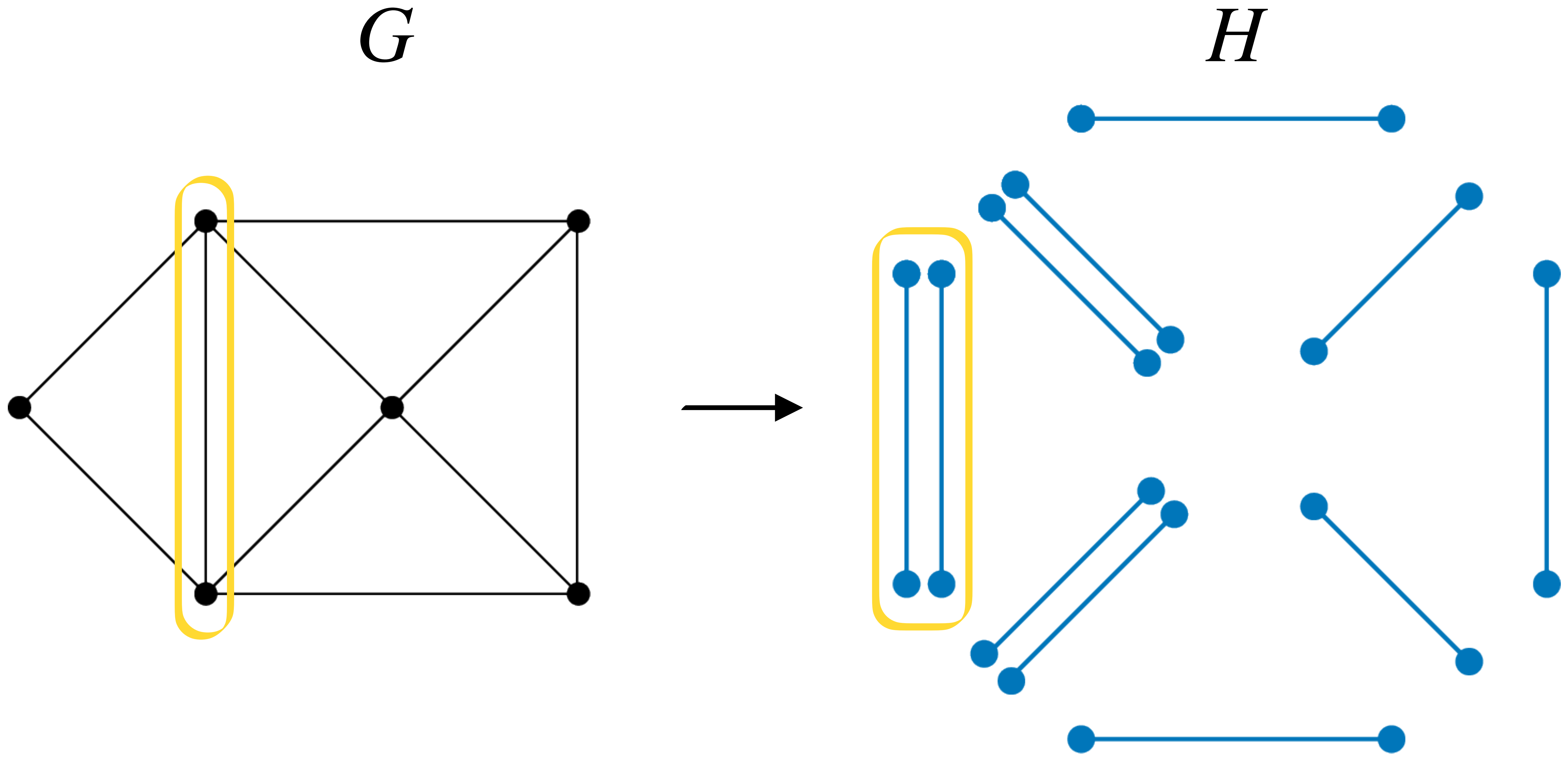

Step 1: For each $uv \in E$, create $d_{uv} := \min\{d(u), d(v)\} - 2$ disjoint edges

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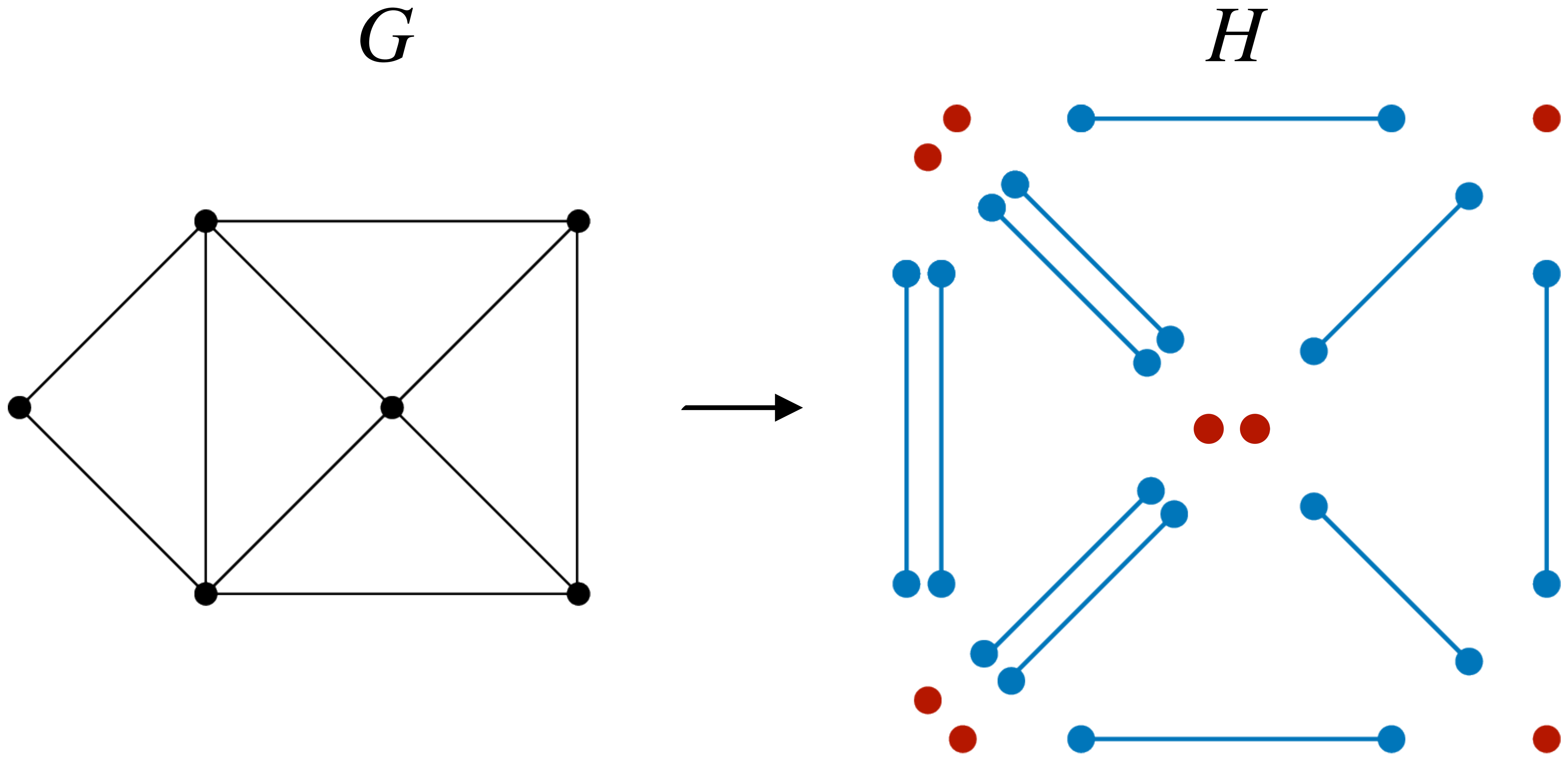

Step ~~2~~: For each vertex v, create $d(v) - 2$ vertices

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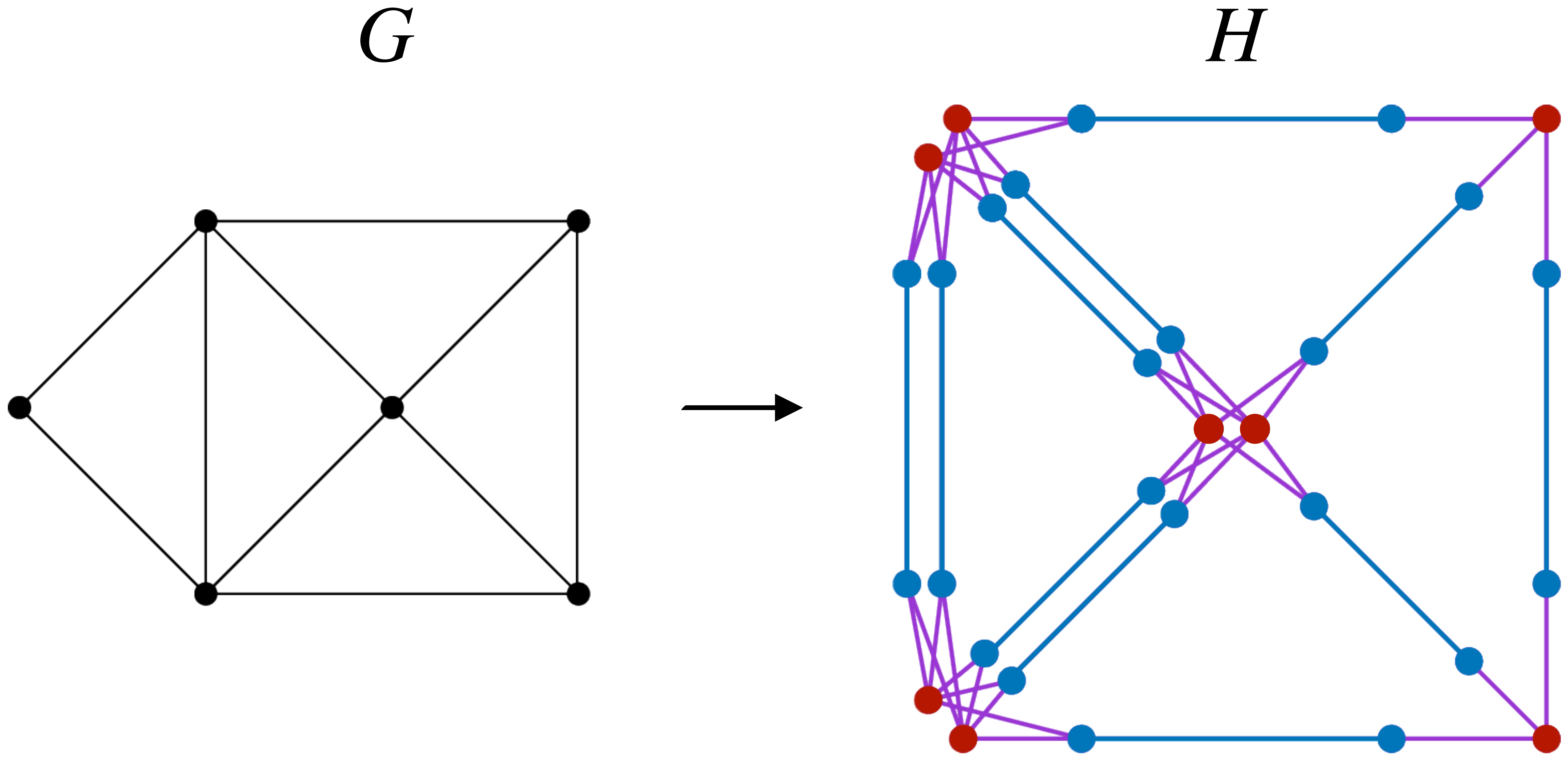
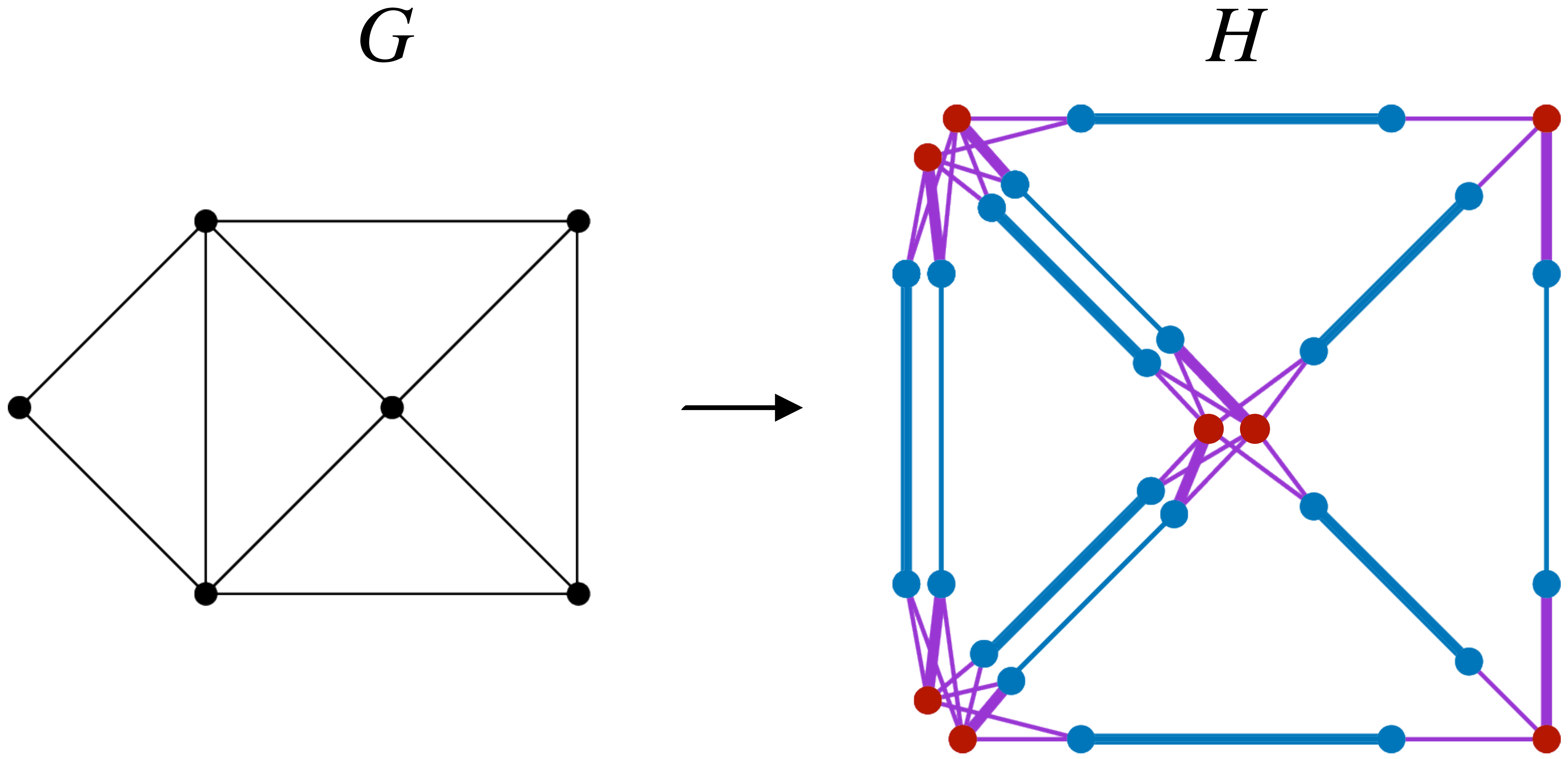


Step 3: Form bicliques at each junction

Approach: Construct a graph H that has a perfect matching if and only if, for every vertex v, we can distribute $d(v) - 2$ units among its incident edges.



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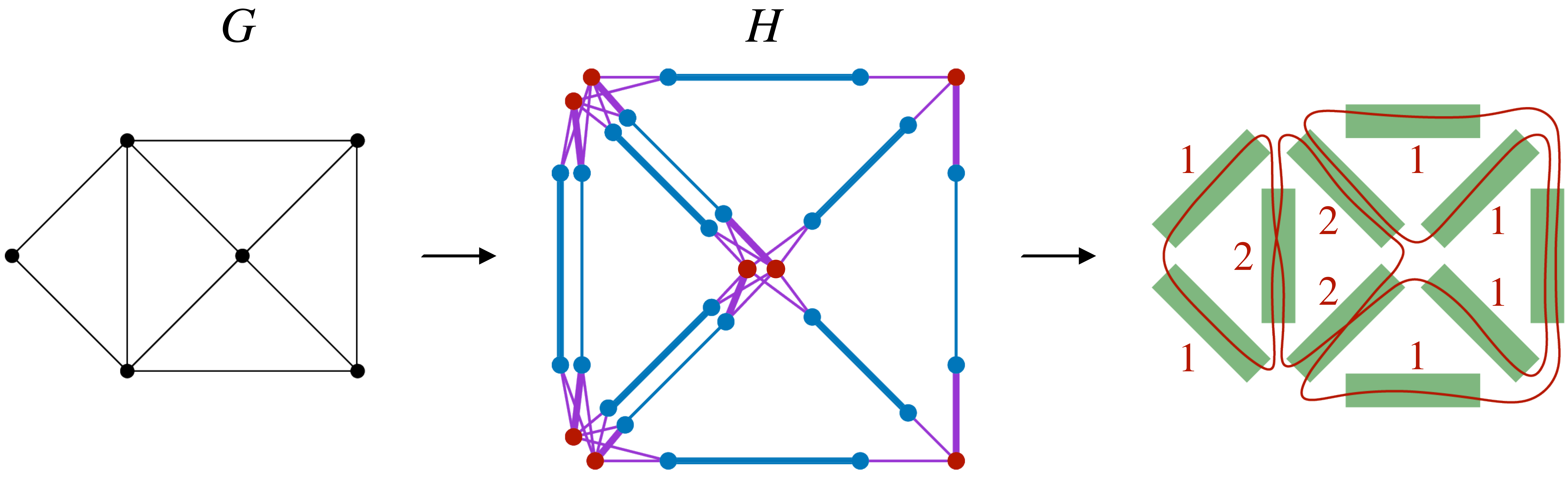

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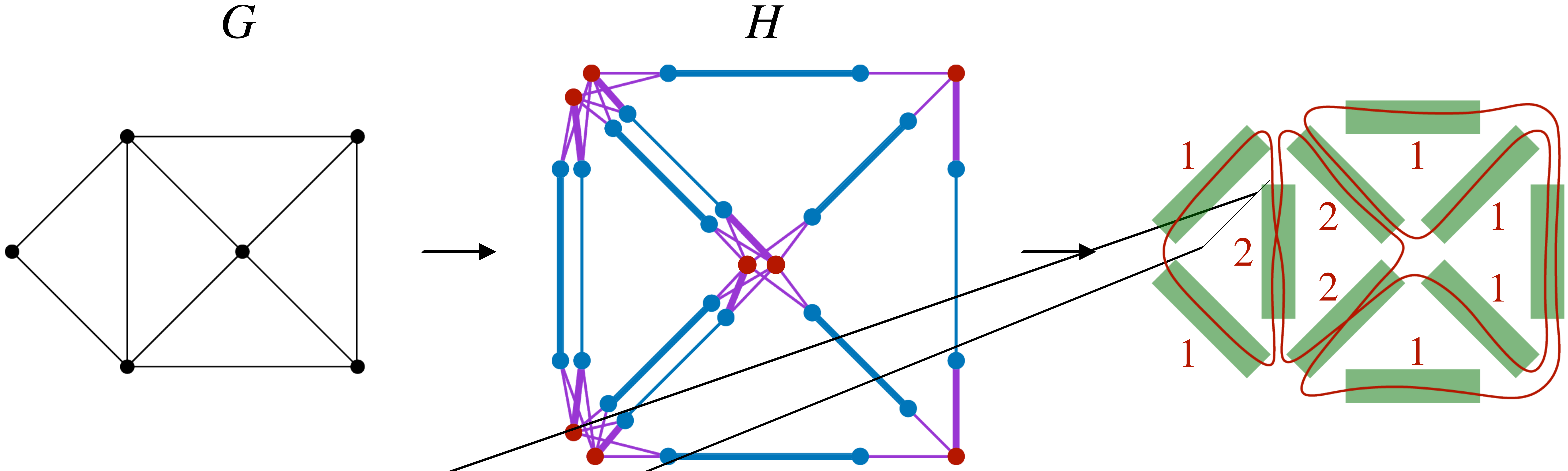

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Theorem. G has a perfect threading if and only if H has a perfect matching.

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